Chapter 2
Pressure Distribution in a Fluid

Motivation. Many fluid problems do not involve motion. They concern the pressure distribution in a static fluid and its effect on solid surfaces and on floating and submerged bodies.

When the fluid velocity is zero, denoted as the *hydrostatic condition*, the pressure variation is due only to the weight of the fluid. Assuming a known fluid in a given gravity field, the pressure may easily be calculated by integration. Important applications in this chapter are (1) pressure distribution in the atmosphere and the oceans, (2) the design of manometer pressure instruments, (3) forces on submerged flat and curved surfaces, (4) buoyancy on a submerged body, and (5) the behavior of floating bodies. The last two result in Archimedes' principles.

If the fluid is moving in *rigid-body motion*, such as a tank of liquid which has been spinning for a long time, the pressure also can be easily calculated, because the fluid is free of shear stress. We apply this idea here to simple rigid-body accelerations in Sec. 2.9. Pressure measurement instruments are discussed in Sec. 2.10. As a matter of fact, pressure also can be easily analyzed in arbitrary (nonrigid-body) motions $\mathbf{V}(x, y, z, t)$, but we defer that subject to Chap. 4.

2.1 Pressure and Pressure Gradient

In Fig. 1.1 we saw that a fluid at rest cannot support shear stress and thus Mohr's circle reduces to a point. In other words, the normal stress on any plane through a fluid element at rest is equal to a unique value called the *fluid pressure* p, taken positive for compression by common convention. This is such an important concept that we shall review it with another approach.

Figure 2.1 shows a small wedge of fluid at rest of size Δx by Δz by Δs and depth b into the paper. There is no shear by definition, but we postulate that the pressures p_x, p_z, and p_n may be different on each face. The weight of the element also may be important. Summation of forces must equal zero (no acceleration) in both the x and z directions.

$$\sum F_x = 0 = p_x b\,\Delta z - p_n b\,\Delta s \sin\theta$$
$$\sum F_z = 0 = p_z b\,\Delta x - p_n b\,\Delta s \cos\theta - \tfrac{1}{2}\gamma b\,\Delta x\,\Delta z \qquad (2.1)$$

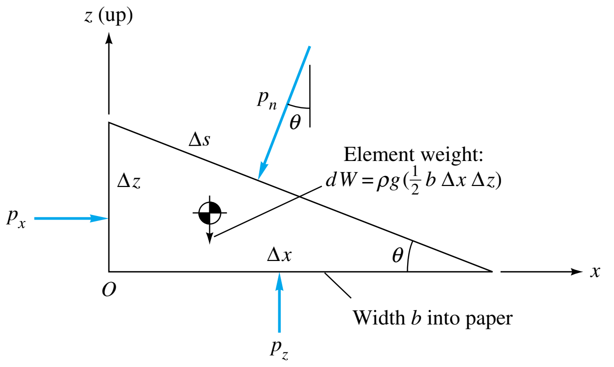


Fig. 2.1 Equilibrium of a small wedge of fluid at rest.

but the geometry of the wedge is such that

$$\Delta s \sin \theta = \Delta z \qquad \Delta s \cos \theta = \Delta x \tag{2.2}$$

Substitution into Eq. (2.1) and rearrangement give

$$p_x = p_n \qquad p_z = p_n + \tfrac{1}{2}\gamma\, \Delta z \tag{2.3}$$

These relations illustrate two important principles of the hydrostatic, or shear-free, condition: (1) There is no pressure change in the horizontal direction, and (2) there is a vertical change in pressure proportional to the density, gravity, and depth change. We shall exploit these results to the fullest, starting in Sec. 2.3.

In the limit as the fluid wedge shrinks to a "point," $\Delta z \to 0$ and Eqs. (2.3) become

$$p_x = p_z = p_n = p \tag{2.4}$$

Since θ is arbitrary, we conclude that the pressure p at a point in a static fluid is independent of orientation.

What about the pressure at a point in a moving fluid? If there are strain rates in a moving fluid, there will be viscous stresses, both shear and normal in general (Sec. 4.3). In that case (Chap. 4) the pressure is defined as the average of the three normal stresses σ_{ii} on the element

$$p = -\tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{2.5}$$

The minus sign occurs because a compression stress is considered to be negative whereas p is positive. Equation (2.5) is subtle and rarely needed since the great majority of viscous flows have negligible viscous normal stresses (Chap. 4).

Pressure Force on a Fluid Element

Pressure (or any other stress, for that matter) causes no net force on a fluid element unless it varies *spatially*.[1] To see this, consider the pressure acting on the two x faces in Fig. 2.2. Let the pressure vary arbitrarily

$$p = p(x, y, z, t) \tag{2.6}$$

[1]An interesting application for a large element is in Fig. 3.7.

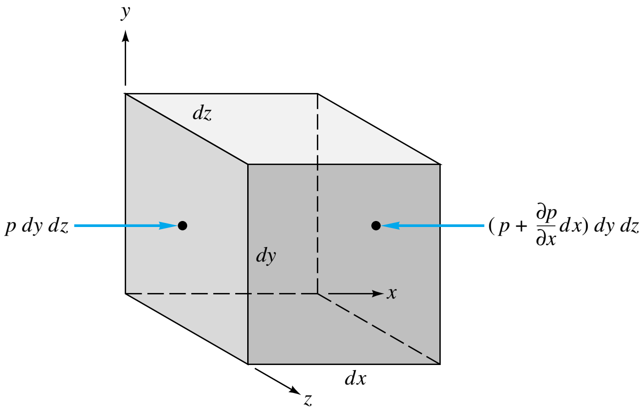


Fig. 2.2 Net x force on an element due to pressure variation.

The net force in the x direction on the element in Fig. 2.2 is given by

$$dF_x = p\ dy\ dz - \left(p + \frac{\partial p}{\partial x}\,dx\right) dy\ dz = -\frac{\partial p}{\partial x}\,dx\ dy\ dz \qquad (2.7)$$

In like manner the net force dF_y involves $-\partial p/\partial y$, and the net force dF_z concerns $-\partial p/\partial z$. The total net-force vector on the element due to pressure is

$$d\mathbf{F}_{\text{press}} = \left(-\mathbf{i}\,\frac{\partial p}{\partial x} - \mathbf{j}\,\frac{\partial p}{\partial y} - \mathbf{k}\,\frac{\partial p}{\partial z}\right) dx\ dy\ dz \qquad (2.8)$$

We recognize the term in parentheses as the negative vector gradient of p. Denoting $\mathbf{f}$ as the net force per unit element volume, we rewrite Eq. (2.8) as

$$\mathbf{f}_{\text{press}} = -\nabla p \qquad (2.9)$$

Thus it is not the pressure but the pressure *gradient* causing a net force which must be balanced by gravity or acceleration or some other effect in the fluid.

2.2 Equilibrium of a Fluid Element

The pressure gradient is a *surface* force which acts on the sides of the element. There may also be a *body* force, due to electromagnetic or gravitational potentials, acting on the entire mass of the element. Here we consider only the gravity force, or weight of the element

$$d\mathbf{F}_{\text{grav}} = \rho\mathbf{g}\ dx\ dy\ dz$$

or

$$\mathbf{f}_{\text{grav}} = \rho\mathbf{g} \qquad (2.10)$$

In general, there may also be a surface force due to the gradient, if any, of the viscous stresses. For completeness, we write this term here without derivation and consider it more thoroughly in Chap. 4. For an incompressible fluid with constant viscosity, the net viscous force is

$$\mathbf{f}_{\text{VS}} = \mu\left(\frac{\partial^2\mathbf{V}}{\partial x^2} + \frac{\partial^2\mathbf{V}}{\partial y^2} + \frac{\partial^2\mathbf{V}}{\partial z^2}\right) = \mu\nabla^2\mathbf{V} \qquad (2.11)$$

where VS stands for viscous stresses and μ is the coefficient of viscosity from Chap. 1. Note that the term $\mathbf{g}$ in Eq. (2.10) denotes the acceleration of gravity, a vector acting toward the center of the earth. On earth the average magnitude of **g** is 32.174 ft/s^2 = 9.807 m/s^2.

The total vector resultant of these three forces—pressure, gravity, and viscous stress—must either keep the element in equilibrium or cause it to move with acceleration **a**. From Newton's law, Eq. (1.2), we have

$$\rho \mathbf{a} = \sum \mathbf{f} = \mathbf{f}_{press} + \mathbf{f}_{grav} + \mathbf{f}_{visc} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \tag{2.12}$$

This is one form of the differential momentum equation for a fluid element, and it is studied further in Chap. 4. Vector addition is implied by Eq. (2.12): The acceleration reflects the local balance of forces and is not necessarily parallel to the local-velocity vector, which reflects the direction of motion at that instant.

This chapter is concerned with cases where the velocity and acceleration are known, leaving one to solve for the pressure variation in the fluid. Later chapters will take up the more general problem where pressure, velocity, and acceleration are all unknown. Rewrite Eq. (2.12) as

$$\nabla p = \rho(\mathbf{g} - \mathbf{a}) + \mu \nabla^2 \mathbf{V} = \mathbf{B}(x, y, z, t) \tag{2.13}$$

where **B** is a short notation for the vector sum on the right-hand side. If **V** and $\mathbf{a} = d\mathbf{V}/dt$ are known functions of space and time and the density and viscosity are known, we can solve Eq. (2.13) for $p(x, y, z, t)$ by direct integration. By components, Eq. (2.13) is equivalent to three simultaneous first-order differential equations

$$\frac{\partial p}{\partial x} = B_x(x, y, z, t) \qquad \frac{\partial p}{\partial y} = B_y(x, y, z, t) \qquad \frac{\partial p}{\partial z} = B_z(x, y, z, t) \tag{2.14}$$

Since the right-hand sides are known functions, they can be integrated systematically to obtain the distribution $p(x, y, z, t)$ except for an unknown function of time, which remains because we have no relation for $\partial p/\partial t$. This extra function is found from a condition of known time variation $p_0(t)$ at some point (x_0, y_0, z_0). If the flow is steady (independent of time), the unknown function is a constant and is found from knowledge of a single known pressure p_0 at a point (x_0, y_0, z_0). If this sounds complicated, it is not; we shall illustrate with many examples. Finding the pressure distribution from a known velocity distribution is one of the easiest problems in fluid mechanics, which is why we put it in Chap. 2.

Examining Eq. (2.13), we can single out at least four special cases:

1. **Flow at rest or at constant velocity:** The acceleration and viscous terms vanish identically, and p depends only upon gravity and density. This is the *hydrostatic* condition. See Sec. 2.3.
2. **Rigid-body translation and rotation:** The viscous term vanishes identically, and p depends only upon the term $\rho(\mathbf{g} - \mathbf{a})$. See Sec. 2.9.
3. **Irrotational motion ($\nabla \times \mathbf{V} \equiv 0$):** The viscous term vanishes identically, and an exact integral called *Bernoulli's equation* can be found for the pressure distribution. See Sec. 4.9.
4. **Arbitrary viscous motion:** Nothing helpful happens, no general rules apply, but still the integration is quite straightforward. See Sec. 6.4.

Let us consider cases 1 and 2 here.

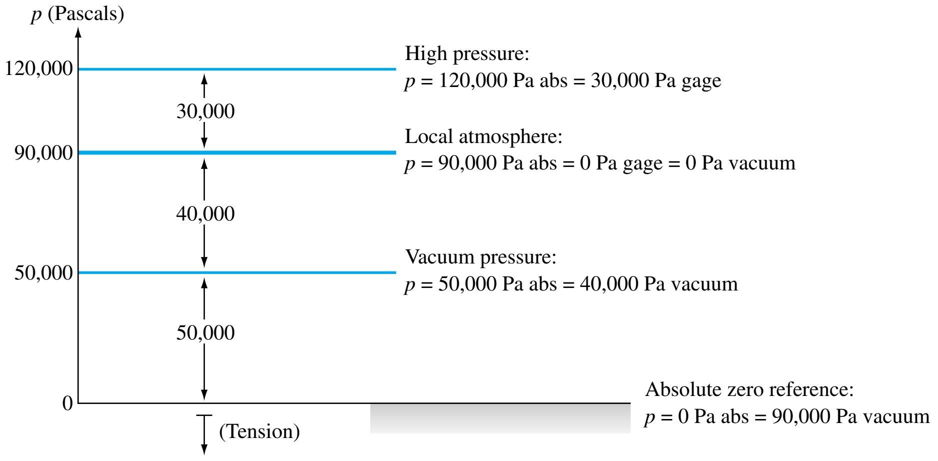


Fig. 2.3 Illustration of absolute, gage, and vacuum pressure readings.

Gage Pressure and Vacuum Pressure: Relative Terms

Before embarking on examples, we should note that engineers are apt to specify pressures as (1) the *absolute* or total magnitude or (2) the value *relative* to the local ambient atmosphere. The second case occurs because many pressure instruments are of *differential* type and record, not an absolute magnitude, but the difference between the fluid pressure and the atmosphere. The measured pressure may be either higher or lower than the local atmosphere, and each case is given a name:

1. $p > p_a$ *Gage* pressure: $p(\text{gage}) = p - p_a$
2. $p < p_a$ *Vacuum* pressure: $p(\text{vacuum}) = p_a - p$

This is a convenient shorthand, and one later adds (or subtracts) atmospheric pressure to determine the absolute fluid pressure.

A typical situation is shown in Fig. 2.3. The local atmosphere is at, say, 90,000 Pa, which might reflect a storm condition in a sea-level location or normal conditions at an altitude of 1000 m. Thus, on this day, p_a = 90,000 Pa absolute = 0 Pa gage = 0 Pa vacuum. Suppose gage 1 in a laboratory reads p_1 = 120,000 Pa absolute. This value may be reported as a *gage* pressure, p_1 = 120,000 − 90,000 = 30,000 Pa *gage*. (One must also record the atmospheric pressure in the laboratory, since p_a changes gradually.) Suppose gage 2 reads p_2 = 50,000 Pa absolute. Locally, this is a *vacuum* pressure and might be reported as p_2 = 90,000 − 50,000 = 40,000 Pa *vacuum*. Occasionally, in the Problems section, we will specify gage or vacuum pressure to keep you alert to this common engineering practice.

2.3 Hydrostatic Pressure Distributions

If the fluid is at rest or at constant velocity, $\mathbf{a} = 0$ and $\nabla^2\mathbf{V} = 0$. Equation (2.13) for the pressure distribution reduces to

$$\nabla p = \rho \mathbf{g} \tag{2.15}$$

This is a *hydrostatic* distribution and is correct for all fluids at rest, regardless of their viscosity, because the viscous term vanishes identically.

Recall from vector analysis that the vector ∇p expresses the magnitude and direction of the maximum spatial rate of increase of the scalar property p. As a result, ∇p

is perpendicular everywhere to surfaces of constant p. Thus Eq. (2.15) states that a fluid in hydrostatic equilibrium will align its constant-pressure surfaces everywhere normal to the local-gravity vector. The maximum pressure increase will be in the direction of gravity, i.e., "down." If the fluid is a liquid, its free surface, being at atmospheric pressure, will be normal to local gravity, or "horizontal." You probably knew all this before, but Eq. (2.15) is the proof of it.

In our customary coordinate system z is "up." Thus the local-gravity vector for small-scale problems is

$$\mathbf{g} = -g\mathbf{k} \tag{2.16}$$

where g is the magnitude of local gravity, for example, 9.807 m/s^2. For these coordinates Eq. (2.15) has the components

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\rho g = -\gamma \tag{2.17}$$

the first two of which tell us that p is independent of x and y. Hence $\partial p/\partial z$ can be replaced by the total derivative dp/dz, and the hydrostatic condition reduces to

$$\frac{dp}{dz} = -\gamma$$

or

$$p_2 - p_1 = -\int_1^2 \gamma \, dz \tag{2.18}$$

Equation (2.18) is the solution to the hydrostatic problem. The integration requires an assumption about the density and gravity distribution. Gases and liquids are usually treated differently.

We state the following conclusions about a hydrostatic condition:

> Pressure in a continuously distributed uniform static fluid varies only with vertical distance and is independent of the shape of the container. The pressure is the same at all points on a given horizontal plane in the fluid. The pressure increases with depth in the fluid.

An illustration of this is shown in Fig. 2.4. The free surface of the container is atmospheric and forms a horizontal plane. Points a, b, c, and d are at equal depth in a horizon-

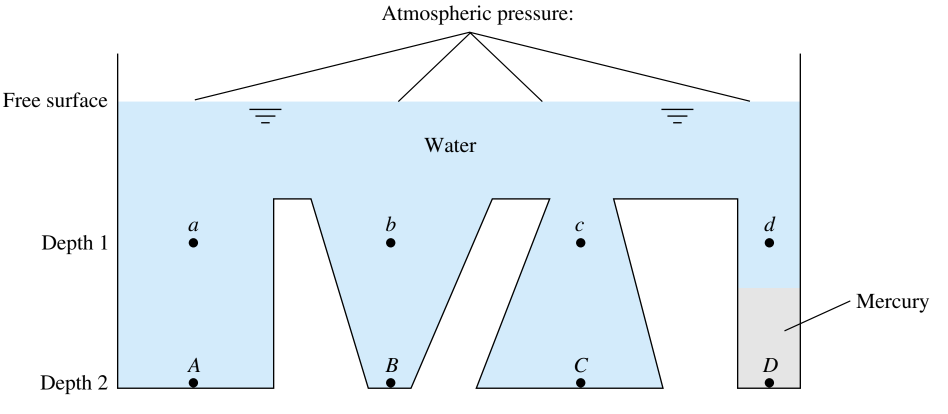


Fig. 2.4 Hydrostatic-pressure distribution. Points a, b, c, and d are at equal depths in water and therefore have identical pressures. Points A, B, and C are also at equal depths in water and have identical pressures higher than a, b, c, and d. Point D has a different pressure from A, B, and C because it is not connected to them by a water path.

tal plane and are interconnected by the same fluid, water; therefore all points have the same pressure. The same is true of points *A*, *B*, and *C* on the bottom, which all have the same higher pressure than at *a*, *b*, *c*, and *d*. However, point *D*, although at the same depth as *A*, *B*, and *C*, has a different pressure because it lies beneath a different fluid, mercury.

Effect of Variable Gravity

For a spherical planet of uniform density, the acceleration of gravity varies inversely as the square of the radius from its center

$$g = g_0\left(\frac{r_0}{r}\right)^2 \tag{2.19}$$

where r_0 is the planet radius and g_0 is the surface value of g. For earth, $r_0 \approx 3960$ statute mi ≈ 6400 km. In typical engineering problems the deviation from r_0 extends from the deepest ocean, about 11 km, to the atmospheric height of supersonic transport operation, about 20 km. This gives a maximum variation in g of $(6400/6420)^2$, or 0.6 percent. We therefore neglect the variation of g in most problems.

Hydrostatic Pressure in Liquids

Liquids are so nearly incompressible that we can neglect their density variation in hydrostatics. In Example 1.7 we saw that water density increases only 4.6 percent at the deepest part of the ocean. Its effect on hydrostatics would be about half of this, or 2.3 percent. Thus we assume constant density in liquid hydrostatic calculations, for which Eq. (2.18) integrates to

Liquids:
$$p_2 - p_1 = -\gamma(z_2 - z_1) \tag{2.20}$$

or
$$z_1 - z_2 = \frac{p_2}{\gamma} - \frac{p_1}{\gamma}$$

We use the first form in most problems. The quantity γ is called the *specific weight* of the fluid, with dimensions of weight per unit volume; some values are tabulated in Table 2.1. The quantity p/γ is a length called the *pressure head* of the fluid.

For lakes and oceans, the coordinate system is usually chosen as in Fig. 2.5, with $z = 0$ at the free surface, where p equals the surface atmospheric pressure p_a. When

Table 2.1 Specific Weight of Some Common Fluids

Fluid	Specific weight γ at 68°F = 20°C, lbf/ft^3	Specific weight γ at 68°F = 20°C, N/m^3
Air (at 1 atm)	0.0752	11.8
Ethyl alcohol	49.2	7,733
SAE 30 oil	55.5	8,720
Water	62.4	9,790
Seawater	64.0	10,050
Glycerin	78.7	12,360
Carbon tetrachloride	99.1	15,570
Mercury	846	133,100

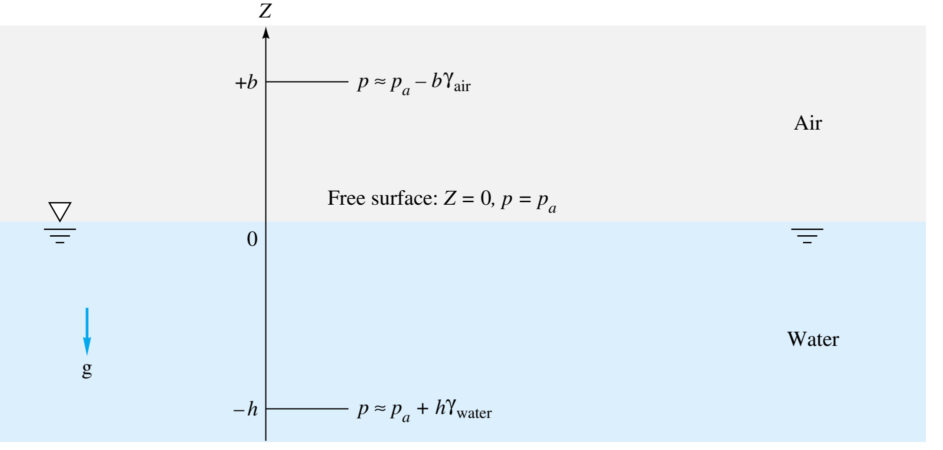


Fig. 2.5 Hydrostatic-pressure distribution in oceans and atmospheres.

we introduce the reference value $(p_1, z_1) = (p_a, 0)$, Eq. (2.20) becomes, for p at any (negative) depth z,

Lakes and oceans: $$p = p_a - \gamma z \tag{2.21}$$

where γ is the average specific weight of the lake or ocean. As we shall see, Eq. (2.21) holds in the atmosphere also with an accuracy of 2 percent for heights z up to 1000 m.

EXAMPLE 2.1

Newfound Lake, a freshwater lake near Bristol, New Hampshire, has a maximum depth of 60 m, and the mean atmospheric pressure is 91 kPa. Estimate the absolute pressure in kPa at this maximum depth.

Solution

From Table 2.1, take $\gamma \approx 9790\ \text{N/m}^3$. With $p_a = 91$ kPa and $z = -60$ m, Eq. (2.21) predicts that the pressure at this depth will be

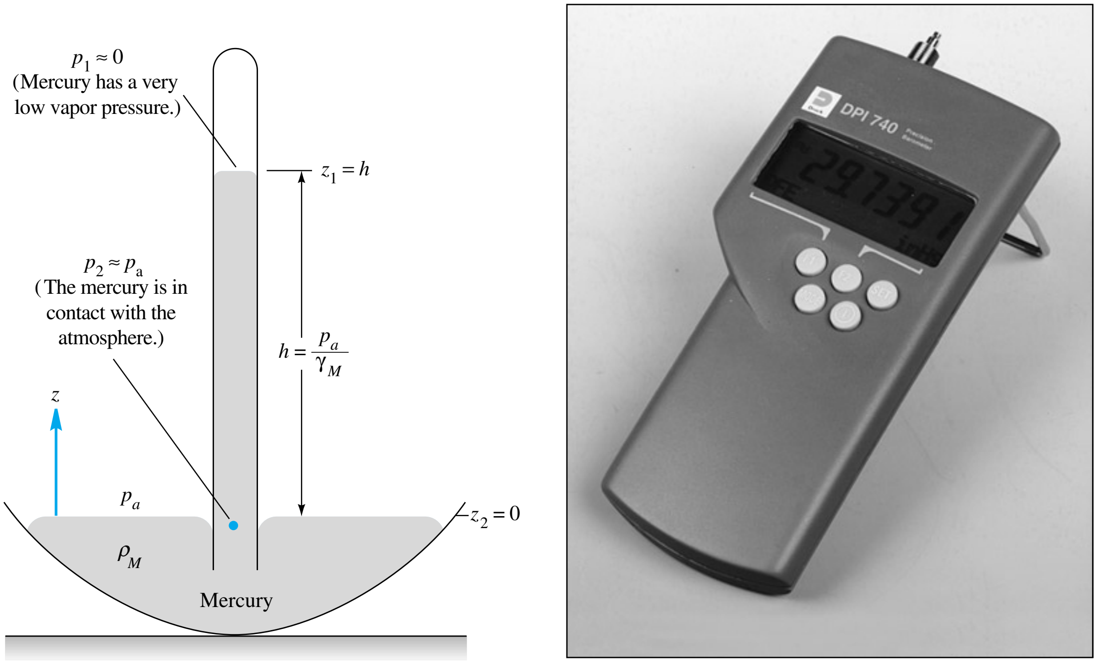
$$p = 91\ \text{kN/m}^2 - (9790\ \text{N/m}^3)(-60\ \text{m})\,\frac{1\ \text{kN}}{1000\ \text{N}}$$

$$= 91\ \text{kPa} + 587\ \text{kN/m}^2 = 678\ \text{kPa} \qquad \textit{Ans.}$$

By omitting p_a we could state the result as $p = 587$ kPa (gage).

The Mercury Barometer

The simplest practical application of the hydrostatic formula (2.20) is the barometer (Fig. 2.6), which measures atmospheric pressure. A tube is filled with mercury and inverted while submerged in a reservoir. This causes a near vacuum in the closed upper end because mercury has an extremely small vapor pressure at room temperatures (0.16 Pa at 20°C). Since atmospheric pressure forces a mercury column to rise a distance h into the tube, the upper mercury surface is at zero pressure.



(a) **(b)**

Fig. 2.6 A barometer measures local absolute atmospheric pressure: (*a*) the height of a mercury column is proportional to p_{atm}; (*b*) a modern portable barometer, with digital readout, uses the resonating silicon element of Fig. 2.28*c*. *(Courtesy of Paul Lupke, Druck Inc.)*

From Fig. 2.6, Eq. (2.20) applies with $p_1 = 0$ at $z_1 = h$ and $p_2 = p_a$ at $z_2 = 0$:

$$p_a - 0 = -\gamma_M(0 - h)$$

or

$$h = \frac{p_a}{\gamma_M} \tag{2.22}$$

At sea-level standard, with $p_a = 101{,}350$ Pa and $\gamma_M = 133{,}100$ N/m^3 from Table 2.1, the barometric height is $h = 101{,}350/133{,}100 = 0.761$ m or 761 mm. In the United States the weather service reports this as an atmospheric "pressure" of 29.96 inHg (inches of mercury). Mercury is used because it is the heaviest common liquid. A water barometer would be 34 ft high.

Hydrostatic Pressure in Gases

Gases are compressible, with density nearly proportional to pressure. Thus density must be considered as a variable in Eq. (2.18) if the integration carries over large pressure changes. It is sufficiently accurate to introduce the perfect-gas law $p = \rho RT$ in Eq. (2.18)

$$\frac{dp}{dz} = -\rho g = -\frac{p}{RT}\, g$$

Separate the variables and integrate between points 1 and 2:

$$\int_1^2 \frac{dp}{p} = \ln \frac{p_2}{p_1} = -\frac{g}{R}\int_1^2 \frac{dz}{T} \tag{2.23}$$

The integral over z requires an assumption about the temperature variation $T(z)$. One common approximation is the *isothermal atmosphere*, where $T = T_0$:

$$p_2 = p_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \tag{2.24}$$

The quantity in brackets is dimensionless. (Think that over; it must be dimensionless, right?) Equation (2.24) is a fair approximation for earth, but actually the earth's mean atmospheric temperature drops off nearly linearly with z up to an altitude of about 36,000 ft (11,000 m):

$$T \approx T_0 - Bz \tag{2.25}$$

Here T_0 is sea-level temperature (absolute) and B is the *lapse rate*, both of which vary somewhat from day to day. By international agreement [1] the following standard values are assumed to apply from 0 to 36,000 ft:

$$T_0 = 518.69°\text{R} = 288.16\text{ K} = 15°\text{C}$$

$$B = 0.003566°\text{R/ft} = 0.00650\text{ K/m} \tag{2.26}$$

This lower portion of the atmosphere is called the *troposphere.* Introducing Eq. (2.25) into (2.23) and integrating, we obtain the more accurate relation

$$p = p_a\left(1 - \frac{Bz}{T_0}\right)^{g/(RB)} \qquad \text{where } \frac{g}{RB} = 5.26 \text{ (air)} \tag{2.27}$$

in the troposphere, with $z = 0$ at sea level. The exponent $g/(RB)$ is dimensionless (again it must be) and has the standard value of 5.26 for air, with $R = 287\text{ m}^2/(\text{s}^2 \cdot \text{K})$.

The U.S. standard atmosphere [1] is sketched in Fig. 2.7. The pressure is seen to be nearly zero at $z = 30$ km. For tabulated properties see Table A.6.

EXAMPLE 2.2

If sea-level pressure is 101,350 Pa, compute the standard pressure at an altitude of 5000 m, using (*a*) the exact formula and (*b*) an isothermal assumption at a standard sea-level temperature of 15°C. Is the isothermal approximation adequate?

Solution

Part (a) Use absolute temperature in the exact formula, Eq. (2.27):

$$p = p_a\left[1 - \frac{(0.00650\text{ K/m})(5000\text{ m})}{288.16\text{ K}}\right]^{5.26} = (101{,}350\text{ Pa})(0.8872)^{5.26}$$

$$= 101{,}350(0.52388) = 54{,}000\text{ Pa} \qquad \textit{Ans. (a)}$$

This is the standard-pressure result given at $z = 5000$ m in Table A.6.

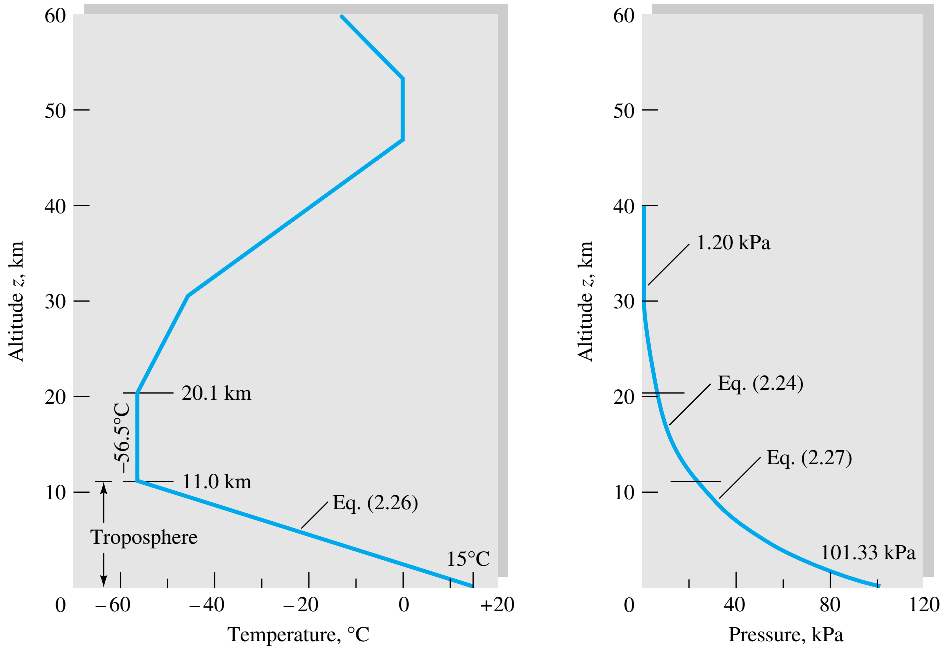


Fig. 2.7 Temperature and pressure distribution in the U.S. standard atmosphere. *(From Ref. 1.)*

Part (b) If the atmosphere were isothermal at 288.16 K, Eq. (2.24) would apply:

$$p \approx p_a \exp\left(-\frac{gz}{RT}\right) = (101{,}350 \text{ Pa}) \exp\left\{-\frac{(9.807 \text{ m/s}^2)(5000 \text{ m})}{[287 \text{ m}^2/(\text{s}^2 \cdot \text{K})](288.16 \text{ K})}\right\}$$

$$= (101{,}350 \text{ Pa}) \exp(-0.5929) \approx 60{,}100 \text{ Pa} \qquad \textit{Ans. (b)}$$

This is 11 percent higher than the exact result. The isothermal formula is inaccurate in the troposphere.

Is the Linear Formula Adequate for Gases?

The linear approximation from Eq. (2.20) or (2.21), $\Delta p \approx \gamma\, \Delta z$, is satisfactory for liquids, which are nearly incompressible. It may be used even over great depths in the ocean. For gases, which are highly compressible, it is valid only over moderate changes in altitude.

The error involved in using the linear approximation (2.21) can be evaluated by expanding the exact formula (2.27) into a series

$$\left(1 - \frac{Bz}{T_0}\right)^n = 1 - n\frac{Bz}{T_0} + \frac{n(n-1)}{2!}\left(\frac{Bz}{T_0}\right)^2 - \cdots \qquad (2.28)$$

where $n = g/(RB)$. Introducing these first three terms of the series into Eq. (2.27) and rearranging, we obtain

$$p = p_a - \gamma_a z\left(1 - \frac{n-1}{2}\frac{Bz}{T_0} + \cdots\right) \qquad (2.29)$$

Thus the error in using the linear formula (2.21) is small if the second term in parentheses in (2.29) is small compared with unity. This is true if

$$z \ll \frac{2T_0}{(n-1)B} = 20{,}800 \text{ m} \tag{2.30}$$

We thus expect errors of less than 5 percent if z or δz is less than 1000 m.

2.4 Application to Manometry

From the hydrostatic formula (2.20), a change in elevation $z_2 - z_1$ of a liquid is equivalent to a change in pressure $(p_2 - p_1)/\gamma$. Thus a static column of one or more liquids or gases can be used to measure pressure differences between two points. Such a device is called a *manometer*. If multiple fluids are used, we must change the density in the formula as we move from one fluid to another. Figure 2.8 illustrates the use of the formula with a column of multiple fluids. The pressure change through each fluid is calculated separately. If we wish to know the total change $p_5 - p_1$, we add the successive changes $p_2 - p_1$, $p_3 - p_2$, $p_4 - p_3$, and $p_5 - p_4$. The intermediate values of p cancel, and we have, for the example of Fig. 2.8,

$$p_5 - p_1 = -\gamma_0(z_2 - z_1) - \gamma_w(z_3 - z_2) - \gamma_G(z_4 - z_3) - \gamma_M(z_5 - z_4) \tag{2.31}$$

No additional simplification is possible on the right-hand side because of the different densities. Notice that we have placed the fluids in order from the lightest on top to the heaviest at bottom. This is the only stable configuration. If we attempt to layer them in any other manner, the fluids will overturn and seek the stable arrangement.

A Memory Device: Up Versus Down

The basic hydrostatic relation, Eq. (2.20), is mathematically correct but vexing to engineers, because it combines two negative signs to have the pressure increase downward. When calculating hydrostatic pressure changes, engineers work instinctively by simply having the pressure increase downward and decrease upward. Thus they use the following mnemonic, or memory, device, first suggested to the writer by Professor John

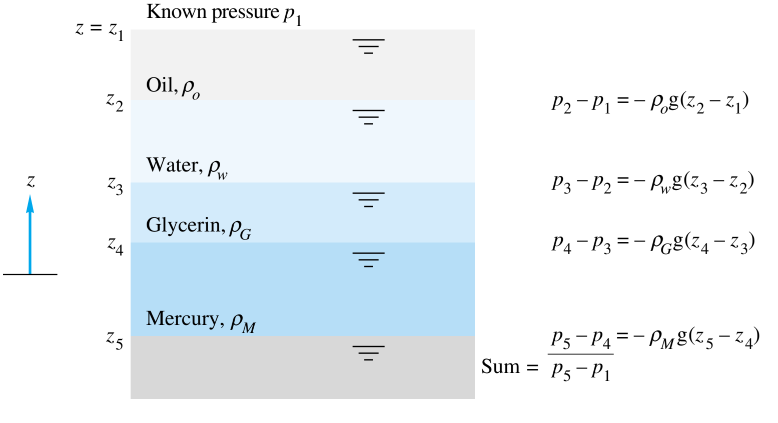


Fig. 2.8 Evaluating pressure changes through a column of multiple fluids.

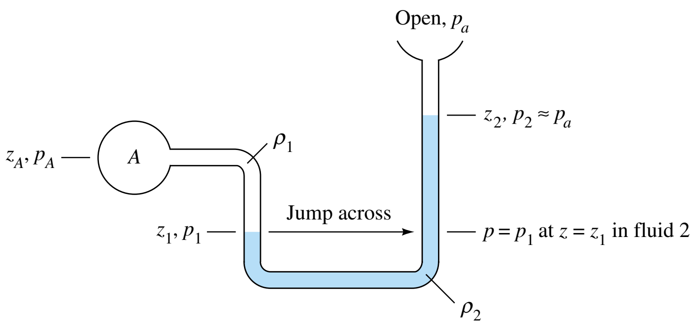


Fig. 2.9 Simple open manometer for measuring p_A relative to atmospheric pressure.

Foss of Michigan State University:

$$p_{down} = p^{up} + \gamma|\Delta z| \tag{2.32}$$

Thus, without worrying too much about which point is "z_1" and which is "z_2", the formula simply increases or decreases the pressure according to whether one is moving down or up. For example, Eq. (2.31) could be rewritten in the following "multiple increase" mode:

$$p_5 = p_1 + \gamma_0|z_1 - z_2| + \gamma_w|z_2 - z_3| + \gamma_G|z_3 - z_4| + \gamma_M|z_4 - z_5|$$

That is, keep adding on pressure increments as you move down through the layered fluid. A different application is a manometer, which involves both "up" and "down" calculations.

Figure 2.9 shows a simple open manometer for measuring p_A in a closed chamber relative to atmospheric pressure p_a, in other words, measuring the gage pressure. The chamber fluid ρ_1 is combined with a second fluid ρ_2, perhaps for two reasons: (1) to protect the environment from a corrosive chamber fluid or (2) because a heavier fluid ρ_2 will keep z_2 small and the open tube can be shorter. One can, of course, apply the basic hydrostatic formula (2.20). Or, more simply, one can begin at A, apply Eq. (2.32) "down" to z_1, jump across fluid 2 (see Fig. 2.9) to the same pressure p_1, and then use Eq. (2.32) "up" to level z_2:

$$p_A + \gamma_1|z_A - z_1| - \gamma_2|z_1 - z_2| = p_2 \approx p_{atm} \tag{2.33}$$

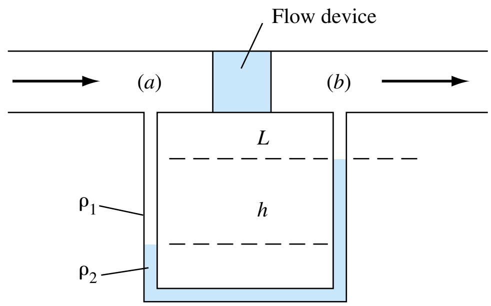
The physical reason that we can "jump across" at section 1 in that a continuous length of the same fluid connects these two equal elevations. The hydrostatic relation (2.20) requires this equality as a form of Pascal's law:

> Any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure.

This idea of jumping across to equal pressures facilitates multiple-fluid problems.

EXAMPLE 2.3

The classic use of a manometer is when two U-tube legs are of equal length, as in Fig. E2.3, and the measurement involves a pressure difference across two horizontal points. The typical ap-



E2.3

plication is to measure pressure change across a flow device, as shown. Derive a formula for the pressure difference $p_a - p_b$ in terms of the system parameters in Fig. E2.3.

Solution

Using our "up-down" concept as in Eq. (2.32), start at (*a*), evaluate pressure changes around the U-tube, and end up at (*b*):

$$p_a + \rho_1 gL + \rho_1 gh - \rho_2 gh - \rho_1 gL = p_b$$

or

$$p_a - p_b = (\rho_2 - \rho_1)gh \qquad \textit{Ans.}$$

The measurement only includes h, the manometer reading. Terms involving L drop out. Note the appearance of the *difference* in densities between manometer fluid and working fluid. It is a common student error to fail to subtract out the working fluid density ρ_1—a serious error if both fluids are liquids and less disastrous numerically if fluid 1 is a gas. Academically, of course, such an error is always considered serious by fluid mechanics instructors.

Although Ex. 2.3, because of its popularity in engineering experiments, is sometimes considered to be the "manometer formula," it is best *not* to memorize it but rather to adapt Eq. (2.20) or (2.32) to each new multiple-fluid hydrostatics problem. For example, Fig. 2.10 illustrates a multiple-fluid manometer problem for finding the

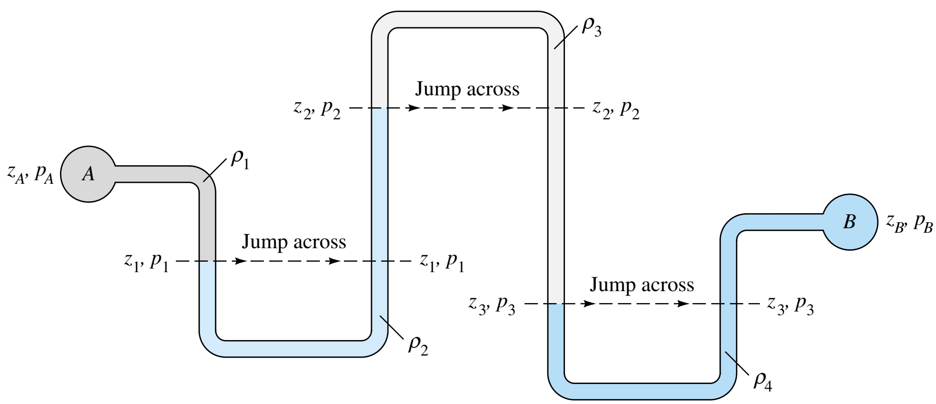


Fig. 2.10 A complicated multiple-fluid manometer to relate p_A to p_B. This system is not especially practical but makes a good homework or examination problem.

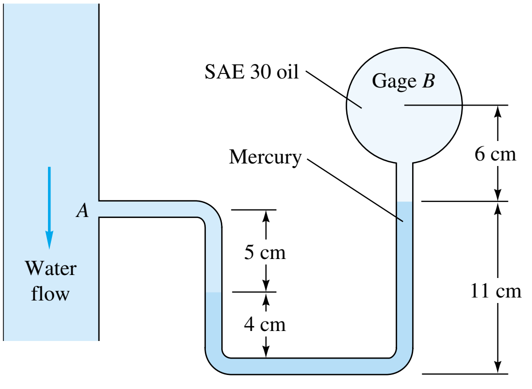
difference in pressure between two chambers A and B. We repeatedly apply Eq. (2.20), jumping across at equal pressures when we come to a continuous mass of the same fluid. Thus, in Fig. 2.10, we compute four pressure differences while making three jumps:

$$p_A - p_B = (p_A - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_B)$$
$$= -\gamma_1(z_A - z_1) - \gamma_2(z_1 - z_2) - \gamma_3(z_2 - z_3) - \gamma_4(z_3 - z_B) \quad (2.34)$$

The intermediate pressures $p_{1,2,3}$ cancel. It looks complicated, but really it is merely *sequential*. One starts at A, goes down to 1, jumps across, goes up to 2, jumps across, goes down to 3, jumps across, and finally goes up to B.

EXAMPLE 2.4

Pressure gage B is to measure the pressure at point A in a water flow. If the pressure at B is 87 kPa, estimate the pressure at A, in kPa. Assume all fluids are at 20°C. See Fig. E2.4.



E2.4

Solution

First list the specific weights from Table 2.1 or Table A.3:

$$\gamma_{water} = 9790\ \text{N/m}^3 \qquad \gamma_{mercury} = 133{,}100\ \text{N/m}^3 \qquad \gamma_{oil} = 8720\ \text{N/m}^3$$

Now proceed from A to B, calculating the pressure change in each fluid and adding:

$$p_A - \gamma_W(\Delta z)_W - \gamma_M(\Delta z)_M - \gamma_O(\Delta z)_O = p_B$$

or $$p_A - (9790\ \text{N/m}^3)(-\,0.05\ \text{m}) - (133{,}100\ \text{N/m}^3)(0.07\ \text{m}) - (8720\ \text{N/m}^3)(0.06\ \text{m})$$

$$= p_A + 489.5\ \text{Pa} - 9317\ \text{Pa} - 523.2\ \text{Pa} = p_B = 87{,}000\ \text{Pa}$$

where we replace N/m^2 by its short name, Pa. The value $\Delta z_M = 0.07$ m is the net elevation change in the mercury (11 cm − 4 cm). Solving for the pressure at point A, we obtain

$$p_A = 96{,}351\ \text{Pa} = 96.4\ \text{kPa} \qquad \textit{Ans.}$$

The intermediate six-figure result of 96,351 Pa is utterly fatuous, since the measurements cannot be made that accurately.

In making these manometer calculations we have neglected the capillary-height changes due to surface tension, which were discussed in Example 1.9. These effects cancel if there is a fluid interface, or *meniscus*, on both sides of the U-tube, as in Fig. 2.9. Otherwise, as in the right-hand U-tube of Fig. 2.10, a capillary correction can be made or the effect can be made negligible by using large-bore ($\geq$ 1 cm) tubes.

2.5 Hydrostatic Forces on Plane Surfaces

A common problem in the design of structures which interact with fluids is the computation of the hydrostatic force on a plane surface. If we neglect density changes in the fluid, Eq. (2.20) applies and the pressure on any submerged surface varies linearly with depth. For a plane surface, the linear stress distribution is exactly analogous to combined bending and compression of a beam in strength-of-materials theory. The hydrostatic problem thus reduces to simple formulas involving the centroid and moments of inertia of the plate cross-sectional area.

Figure 2.11 shows a plane panel of arbitrary shape completely submerged in a liquid. The panel plane makes an arbitrary angle θ with the horizontal free surface, so that the depth varies over the panel surface. If h is the depth to any element area dA of the plate, from Eq. (2.20) the pressure there is $p = p_a + \gamma h$.

To derive formulas involving the plate shape, establish an xy coordinate system in the plane of the plate with the origin at its centroid, plus a dummy coordinate ξ down from the surface in the plane of the plate. Then the total hydrostatic force on one side of the plate is given by

$$F = \int p\, dA = \int (p_a + \gamma h)\, dA = p_a A + \gamma \int h\, dA \tag{2.35}$$

The remaining integral is evaluated by noticing from Fig. 2.11 that $h = \xi \sin\theta$ and,

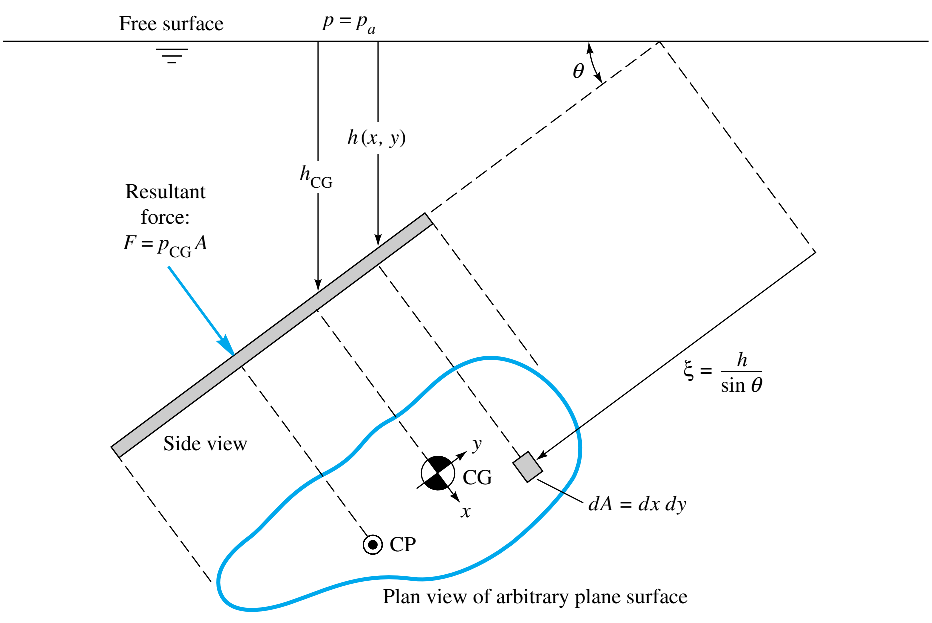


Fig. 2.11 Hydrostatic force and center of pressure on an arbitrary plane surface of area A inclined at an angle θ below the free surface.

by definition, the centroidal slant distance from the surface to the plate is

$$\xi_{CG} = \frac{1}{A} \int \xi \, dA \tag{2.36}$$

Therefore, since θ is constant along the plate, Eq. (2.35) becomes

$$F = p_a A + \gamma \sin \theta \int \xi \, dA = p_a A + \gamma \sin \theta \, \xi_{CG} A \tag{2.37}$$

Finally, unravel this by noticing that $\xi_{CG} \sin \theta = h_{CG}$, the depth straight down from the surface to the plate centroid. Thus

$$F = p_a A + \gamma h_{CG} A = (p_a + \gamma h_{CG}) A = p_{CG} A \tag{2.38}$$

The force on one side of any plane submerged surface in a uniform fluid equals the pressure at the plate centroid times the plate area, independent of the shape of the plate or the angle θ at which it is slanted.

Equation (2.38) can be visualized physically in Fig. 2.12 as the resultant of a linear stress distribution over the plate area. This simulates combined compression and bending of a beam of the same cross section. It follows that the "bending" portion of the stress causes no force if its "neutral axis" passes through the plate centroid of area. Thus the remaining "compression" part must equal the centroid stress times the plate area. This is the result of Eq. (2.38).

However, to balance the bending-moment portion of the stress, the resultant force F does not act through the centroid but below it toward the high-pressure side. Its line of action passes through the *center of pressure* CP of the plate, as sketched in Fig. 2.11. To find the coordinates (x_{CP}, y_{CP}), we sum moments of the elemental force $p\, dA$ about the centroid and equate to the moment of the resultant F. To compute y_{CP}, we equate

$$F y_{CP} = \int y p \, dA = \int y(p_a + \gamma \xi \sin \theta) \, dA = \gamma \sin \theta \int y \xi \, dA \tag{2.39}$$

The term $\int p_a y \, dA$ vanishes by definition of centroidal axes. Introducing $\xi = \xi_{CG} - y$,

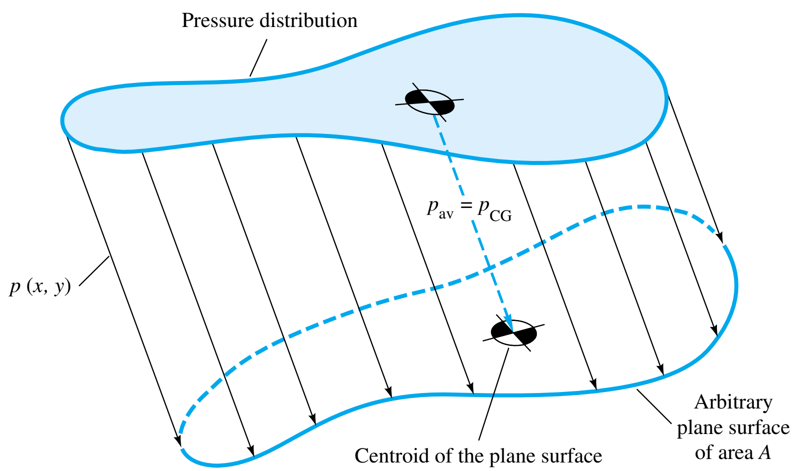


Fig. 2.12 The hydrostatic-pressure force on a plane surface is equal, regardless of its shape, to the resultant of the three-dimensional linear pressure distribution on that surface $F = p_{CG} A$.

we obtain

$$F y_{CP} = \gamma \sin\theta\left(\xi_{CG}\int y\, dA - \int y^2\, dA\right) = -\gamma \sin\theta\, I_{xx} \tag{2.40}$$

where again $\int y\, dA = 0$ and I_{xx} is the area moment of inertia of the plate area about its centroidal x axis, computed in the plane of the plate. Substituting for F gives the result

$$y_{CP} = -\gamma \sin\theta \frac{I_{xx}}{p_{CG} A} \tag{2.41}$$

The negative sign in Eq. (2.41) shows that y_{CP} is below the centroid at a deeper level and, unlike F, depends upon angle θ. If we move the plate deeper, y_{CP} approaches the centroid because every term in Eq. (2.41) remains constant except p_{CG}, which increases.

The determination of x_{CP} is exactly similar:

$$F x_{CP} = \int xp\, dA = \int x[p_a + \gamma(\xi_{CG} - y)\sin\theta]\, dA$$

$$= -\gamma \sin\theta \int xy\, dA = -\gamma \sin\theta\, I_{xy} \tag{2.42}$$

where I_{xy} is the product of inertia of the plate, again computed in the plane of the plate. Substituting for F gives

$$x_{CP} = -\gamma \sin\theta \frac{I_{xy}}{p_{CG} A} \tag{2.43}$$

For positive I_{xy}, x_{CP} is negative because the dominant pressure force acts in the third, or lower left, quadrant of the panel. If $I_{xy} = 0$, usually implying symmetry, $x_{CP} = 0$ and the center of pressure lies directly below the centroid on the y axis.

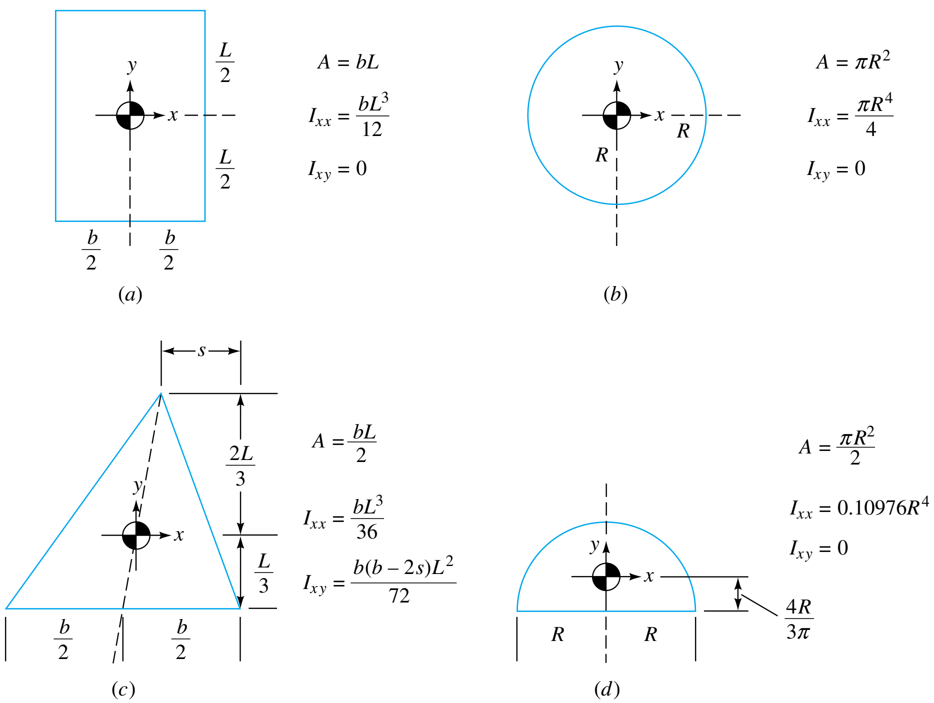


Fig. 2.13 Centroidal moments of inertia for various cross sections: (*a*) rectangle, (*b*) circle, (*c*) triangle, and (*d*) semicircle.

Gage-Pressure Formulas

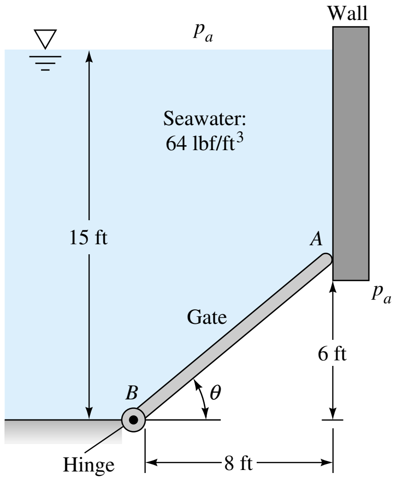
In most cases the ambient pressure p_a is neglected because it acts on both sides of the plate; e.g., the other side of the plate is inside a ship or on the dry side of a gate or dam. In this case $p_{CG} = \gamma h_{CG}$, and the center of pressure becomes independent of specific weight

$$F = \gamma h_{CG} A \qquad y_{CP} = -\frac{I_{xx} \sin \theta}{h_{CG} A} \qquad x_{CP} = -\frac{I_{xy} \sin \theta}{h_{CG} A} \tag{2.44}$$

Figure 2.13 gives the area and moments of inertia of several common cross sections for use with these formulas.

EXAMPLE 2.5

The gate in Fig. E2.5*a* is 5 ft wide, is hinged at point *B*, and rests against a smooth wall at point *A*. Compute (*a*) the force on the gate due to seawater pressure, (*b*) the horizontal force *P* exerted by the wall at point *A*, and (*c*) the reactions at the hinge *B*.



E2.5a

Solution

Part (a) By geometry the gate is 10 ft long from *A* to *B*, and its centroid is halfway between, or at elevation 3 ft above point *B*. The depth h_{CG} is thus 15 − 3 = 12 ft. The gate area is 5(10) = 50 ft^2. Neglect p_a as acting on both sides of the gate. From Eq. (2.38) the hydrostatic force on the gate is

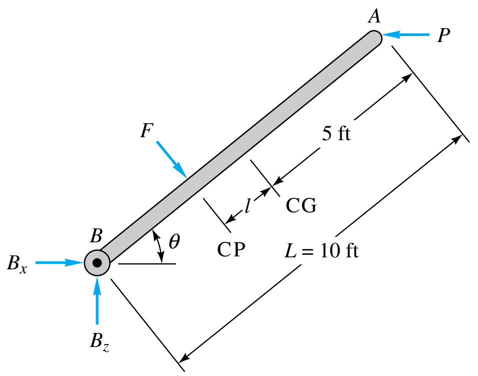
$$F = p_{CG} A = \gamma h_{CG} A = (64 \text{ lbf/ft}^3)(12 \text{ ft})(50 \text{ ft}^2) = 38{,}400 \text{ lbf} \qquad \textit{Ans. (a)}$$

Part (b) First we must find the center of pressure of *F*. A free-body diagram of the gate is shown in Fig. E2.5*b*. The gate is a rectangle, hence

$$I_{xy} = 0 \qquad \text{and} \qquad I_{xx} = \frac{bL^3}{12} = \frac{(5 \text{ ft})(10 \text{ ft})^3}{12} = 417 \text{ ft}^4$$

The distance *l* from the CG to the CP is given by Eq. (2.44) since p_a is neglected.

$$l = -y_{CP} = +\frac{I_{xx} \sin \theta}{h_{CG} A} = \frac{(417 \text{ ft}^4)(\frac{6}{10})}{(12 \text{ ft})(50 \text{ ft}^2)} = 0.417 \text{ ft}$$



E2.5b

The distance from point B to force F is thus $10 - l - 5 = 4.583$ ft. Summing moments counterclockwise about B gives

$$PL \sin \theta - F(5 - l) = P(6 \text{ ft}) - (38{,}400 \text{ lbf})(4.583 \text{ ft}) = 0$$

or $$P = 29{,}300 \text{ lbf}$$ *Ans. (b)*

Part (c) With F and P known, the reactions B_x and B_z are found by summing forces on the gate

$$\sum F_x = 0 = B_x + F \sin \theta - P = B_x + 38{,}400(0.6) - 29{,}300$$

or $$B_x = 6300 \text{ lbf}$$

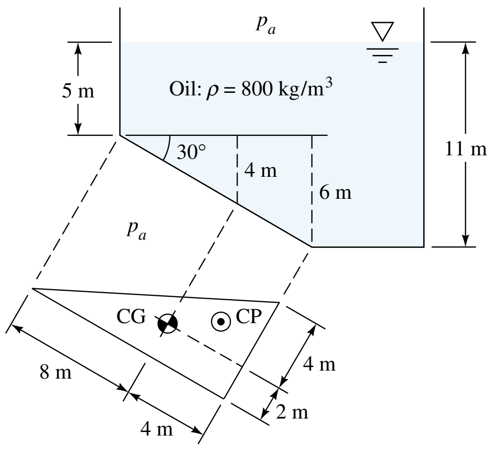
$$\sum F_z = 0 = B_z - F \cos \theta = B_z - 38{,}400(0.8)$$

or $$B_z = 30{,}700 \text{ lbf}$$ *Ans. (c)*

This example should have reviewed your knowledge of statics.

EXAMPLE 2.6

A tank of oil has a right-triangular panel near the bottom, as in Fig. E2.6. Omitting p_a, find the (*a*) hydrostatic force and (*b*) CP on the panel.



E2.6

Solution

Part (a) The triangle has properties given in Fig. 2.13*c*. The centroid is one-third up (4 m) and one-third over (2 m) from the lower left corner, as shown. The area is

$$\tfrac{1}{2}(6\text{ m})(12\text{ m}) = 36\text{ m}^2$$

The moments of inertia are

$$I_{xx} = \frac{bL^3}{36} = \frac{(6\text{ m})(12\text{ m})^3}{36} = 288\text{ m}^4$$

and

$$I_{xy} = \frac{b(b-2s)L^2}{72} = \frac{(6\text{ m})[6\text{ m} - 2(6\text{ m})](12\text{ m})^2}{72} = -72\text{ m}^4$$

The depth to the centroid is $h_{CG} = 5 + 4 = 9$ m; thus the hydrostatic force from Eq. (2.44) is

$$F = \rho g h_{CG} A = (800\text{ kg/m}^3)(9.807\text{ m/s}^2)(9\text{ m})(36\text{ m}^2)$$

$$= 2.54 \times 10^6\ (\text{kg}\cdot\text{m})/\text{s}^2 = 2.54 \times 10^6\text{ N} = 2.54\text{ MN} \qquad \textit{Ans. (a)}$$

Part (b) The CP position is given by Eqs. (2.44):

$$y_{CP} = -\frac{I_{xx}\sin\theta}{h_{CG}A} = -\frac{(288\text{ m}^4)(\sin 30°)}{(9\text{ m})(36\text{ m}^2)} = -0.444\text{ m}$$

$$x_{CP} = -\frac{I_{xy}\sin\theta}{h_{CG}A} = -\frac{(-72\text{ m}^4)(\sin 30°)}{(9\text{ m})(36\text{ m}^2)} = +0.111\text{ m} \qquad \textit{Ans. (b)}$$

The resultant force $F = 2.54$ MN acts through this point, which is down and to the right of the centroid, as shown in Fig. E2.6.

2.6 Hydrostatic Forces on Curved Surfaces

The resultant pressure force on a curved surface is most easily computed by separating it into horizontal and vertical components. Consider the arbitrary curved surface sketched in Fig. 2.14*a*. The incremental pressure forces, being normal to the local area element, vary in direction along the surface and thus cannot be added numerically. We

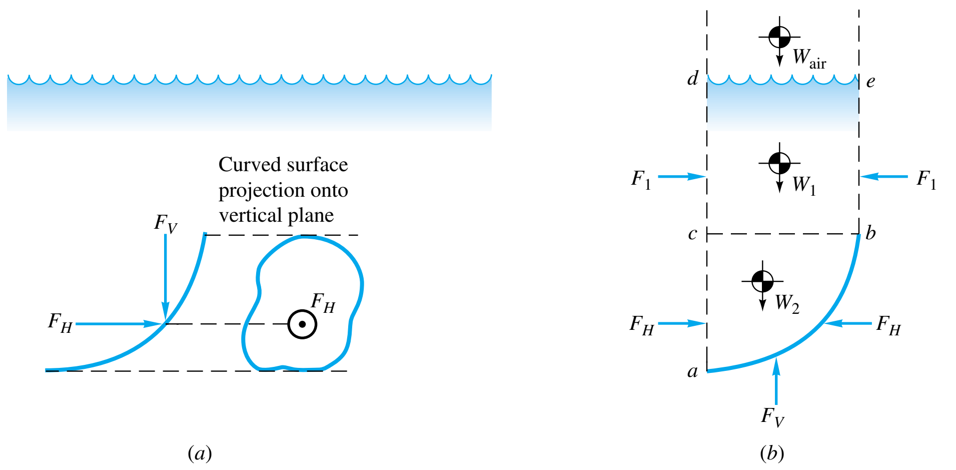


Fig. 2.14 Computation of hydrostatic force on a curved surface: (*a*) submerged curved surface; (*b*) free-body diagram of fluid above the curved surface.

could sum the separate three components of these elemental pressure forces, but it turns out that we need not perform a laborious three-way integration.

Figure 2.14*b* shows a free-body diagram of the column of fluid contained in the vertical projection above the curved surface. The desired forces F_H and F_V are exerted by the surface on the fluid column. Other forces are shown due to fluid weight and horizontal pressure on the vertical sides of this column. The column of fluid must be in static equilibrium. On the upper part of the column *bcde*, the horizontal components F_1 exactly balance and are not relevant to the discussion. On the lower, irregular portion of fluid *abc* adjoining the surface, summation of horizontal forces shows that the desired force F_H due to the curved surface is exactly equal to the force F_H on the vertical left side of the fluid column. This left-side force can be computed by the plane-surface formula, Eq. (2.38), based on a vertical projection of the area of the curved surface. This is a general rule and simplifies the analysis:

> The horizontal component of force on a curved surface equals the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component.

If there are two horizontal components, both can be computed by this scheme.

Summation of vertical forces on the fluid free body then shows that

$$F_V = W_1 + W_2 + W_{air} \tag{2.45}$$

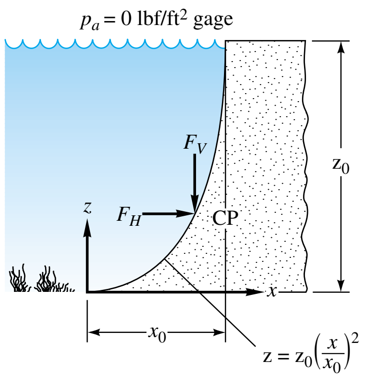
We can state this in words as our second general rule:

> The vertical component of pressure force on a curved surface equals in magnitude and direction the weight of the entire column of fluid, both liquid and atmosphere, above the curved surface.

Thus the calculation of F_V involves little more than finding centers of mass of a column of fluid—perhaps a little integration if the lower portion *abc* has a particularly vexing shape.

EXAMPLE 2.7

A dam has a parabolic shape $z/z_0 = (x/x_0)^2$ as shown in Fig. E2.7*a*, with x_0 = 10 ft and z_0 = 24 ft. The fluid is water, γ = 62.4 lbf/ft^3, and atmospheric pressure may be omitted. Compute the



E2.7a

forces F_H and F_V on the dam and the position CP where they act. The width of the dam is 50 ft.

Solution

The vertical projection of this curved surface is a rectangle 24 ft high and 50 ft wide, with its centroid halfway down, or $h_{CG} = 12$ ft. The force F_H is thus

$$F_H = \gamma h_{CG} A_{proj} = (62.4 \text{ lbf/ft}^3)(12 \text{ ft})(24 \text{ ft})(50 \text{ ft})$$

$$= 899{,}000 \text{ lbf} = 899 \times 10^3 \text{ lbf} \qquad \textit{Ans.}$$

The line of action of F_H is below the centroid by an amount

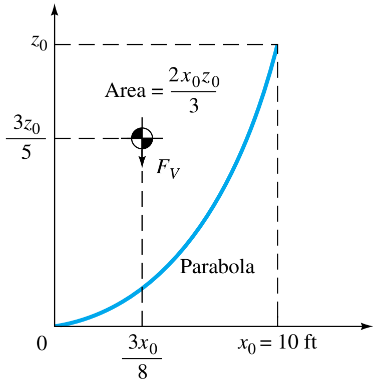
$$y_{CP} = -\frac{I_{xx} \sin\theta}{h_{CG} A_{proj}} = -\frac{\frac{1}{12}(50 \text{ ft})(24 \text{ ft})^3(\sin 90°)}{(12 \text{ ft})(24 \text{ ft})(50 \text{ ft})} = -4 \text{ ft}$$

Thus F_H is 12 + 4 = 16 ft, or two-thirds, down from the free surface or 8 ft from the bottom, as might have been evident by inspection of the triangular pressure distribution.

The vertical component F_V equals the weight of the parabolic portion of fluid above the curved surface. The geometric properties of a parabola are shown in Fig. E2.7*b*. The weight of this amount of water is

$$F_V = \gamma(\tfrac{2}{3}x_0 z_0 b) = (62.4 \text{ lbf/ft}^3)(\tfrac{2}{3})(10 \text{ ft})(24 \text{ ft})(50 \text{ ft})$$

$$= 499{,}000 \text{ lbf} = 499 \times 10^3 \text{ lbf} \qquad \textit{Ans.}$$



E2.7b

This acts downward on the surface at a distance $3x_0/8 = 3.75$ ft over from the origin of coordinates. Note that the vertical distance $3z_0/5$ in Fig. E2.7*b* is irrelevant.

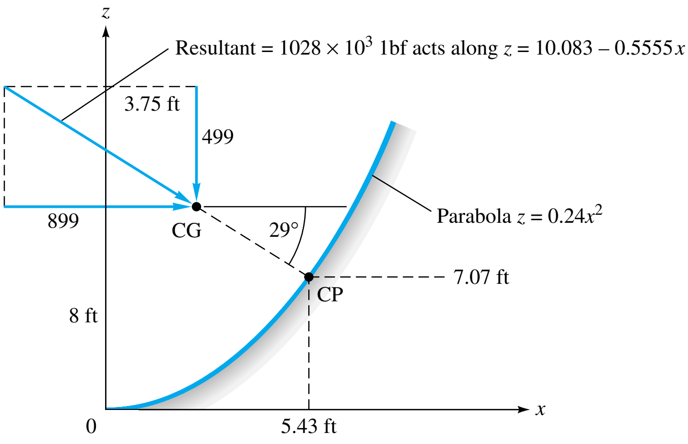
The total resultant force acting on the dam is

$$F = (F_H^2 + F_V^2)^{1/2} = [(499)^2 + (899)^2]^{1/2} = 1028 \times 10^3 \text{ lbf}$$

As seen in Fig. E2.7*c*, this force acts down and to the right at an angle of $29° = \tan^{-1} \frac{499}{899}$. The force F passes through the point $(x, z) = (3.75$ ft, 8 ft). If we move down along the 29° line until we strike the dam, we find an equivalent center of pressure on the dam at

$$x_{CP} = 5.43 \text{ ft} \qquad z_{CP} = 7.07 \text{ ft} \qquad \textit{Ans.}$$

This definition of CP is rather artificial, but this is an unavoidable complication of dealing with a curved surface.



E2.7c

2.7 Hydrostatic Forces in Layered Fluids

The formulas for plane and curved surfaces in Secs. 2.5 and 2.6 are valid only for a fluid of uniform density. If the fluid is layered with different densities, as in Fig. 2.15, a single formula cannot solve the problem because the slope of the linear pressure distribution changes between layers. However, the formulas apply separately to each layer, and thus the appropriate remedy is to compute and sum the separate layer forces and moments.

Consider the slanted plane surface immersed in a two-layer fluid in Fig. 2.15. The slope of the pressure distribution becomes steeper as we move down into the denser

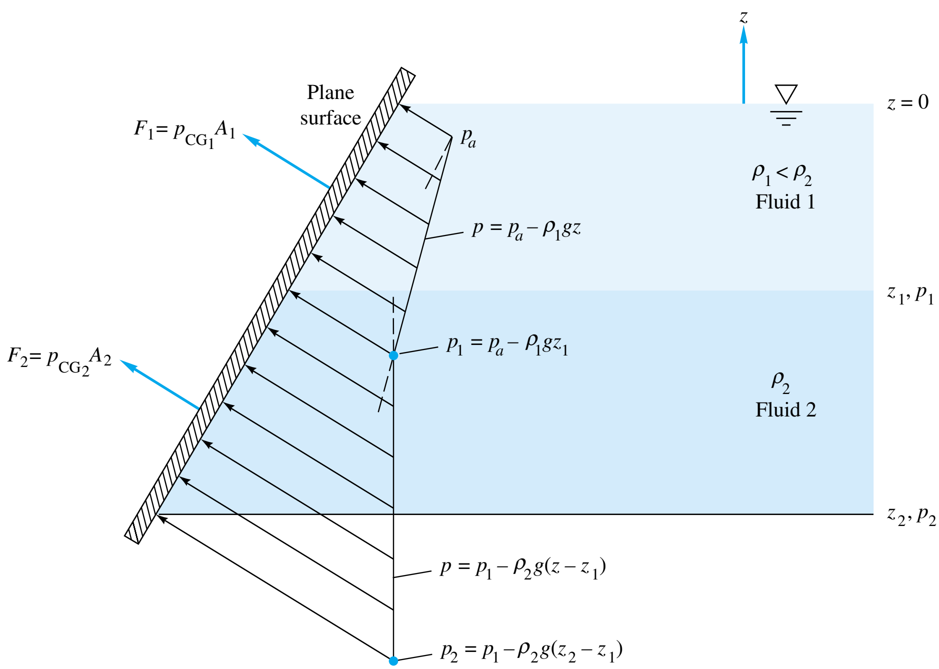


Fig. 2.15 Hydrostatic forces on a surface immersed in a layered fluid must be summed in separate pieces.

second layer. The total force on the plate does *not* equal the pressure at the centroid times the plate area, but the plate portion in each layer does satisfy the formula, so that we can sum forces to find the total:

$$F = \sum F_i = \sum p_{CG_i} A_i \qquad (2.46)$$

Similarly, the centroid of the plate portion in each layer can be used to locate the center of pressure on that portion

$$y_{CP_i} = -\frac{\rho_i g \sin \theta_i I_{xx_i}}{p_{CG_i} A_i} \qquad x_{CP_i} = -\frac{\rho_i g \sin \theta_i I_{xy_i}}{p_{CG_i} A_i} \qquad (2.47)$$

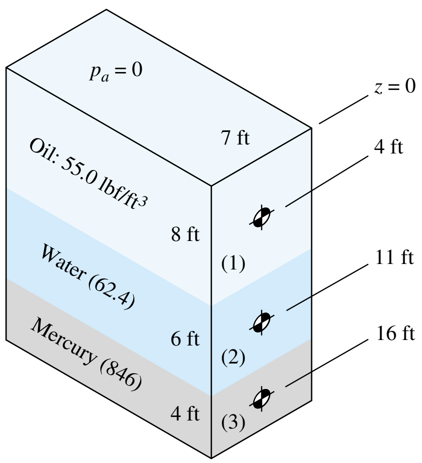
These formulas locate the center of pressure of that particular F_i with respect to the centroid of that particular portion of plate in the layer, not with respect to the centroid of the entire plate. The center of pressure of the total force $F = \sum F_i$ can then be found by summing moments about some convenient point such as the surface. The following example will illustrate.

EXAMPLE 2.8

A tank 20 ft deep and 7 ft wide is layered with 8 ft of oil, 6 ft of water, and 4 ft of mercury. Compute (*a*) the total hydrostatic force and (*b*) the resultant center of pressure of the fluid on the right-hand side of the tank.

Solution

Part (a) Divide the end panel into three parts as sketched in Fig. E2.8, and find the hydrostatic pressure at the centroid of each part, using the relation (2.38) in steps as in Fig. E2.8:



E2.8

$$p_{CG_1} = (55.0 \text{ lbf/ft}^3)(4 \text{ ft}) = 220 \text{ lbf/ft}^2$$

$$p_{CG_2} = (55.0)(8) + 62.4(3) = 627 \text{ lbf/ft}^2$$

$$p_{CG_3} = (55.0)(8) + 62.4(6) + 846(2) = 2506 \text{ lbf/ft}^2$$

These pressures are then multiplied by the respective panel areas to find the force on each portion:

$$F_1 = p_{CG_1}A_1 = (220 \text{ lbf/ft}^2)(8 \text{ ft})(7 \text{ ft}) = 12{,}300 \text{ lbf}$$

$$F_2 = p_{CG_2}A_2 = 627(6)(7) = 26{,}300 \text{ lbf}$$

$$F_3 = p_{CG_3}A_3 = 2506(4)(7) = \underline{70{,}200 \text{ lbf}}$$

$$F = \sum F_i = 108{,}800 \text{ lbf} \qquad \textit{Ans. (a)}$$

Part (b) Equations (2.47) can be used to locate the CP of each force F_i, noting that $\theta = 90°$ and $\sin\theta = 1$ for all parts. The moments of inertia are $I_{xx_1} = (7 \text{ ft})(8 \text{ ft})^3/12 = 298.7 \text{ ft}^4$, $I_{xx_2} = 7(6)^3/12 = 126.0 \text{ ft}^4$, and $I_{xx_3} = 7(4)^3/12 = 37.3 \text{ ft}^4$. The centers of pressure are thus at

$$y_{CP_1} = -\frac{\rho_1 g I_{xx_1}}{F_1} = -\frac{(55.0 \text{ lbf/ft}^3)(298.7 \text{ ft}^4)}{12{,}300 \text{ lbf}} = -1.33 \text{ ft}$$

$$y_{CP_2} = -\frac{62.4(126.0)}{26{,}300} = -0.30 \text{ ft} \qquad y_{CP_3} = -\frac{846(37.3)}{70{,}200} = -0.45 \text{ ft}$$

This locates $z_{CP_1} = -4 - 1.33 = -5.33$ ft, $z_{CP_2} = -11 - 0.30 = -11.30$ ft, and $z_{CP_3} = -16 - 0.45 = -16.45$ ft. Summing moments about the surface then gives

$$\sum F_i z_{CP_i} = F z_{CP}$$

or

$$12{,}300(-5.33) + 26{,}300(-11.30) + 70{,}200(-16.45) = 108{,}800 z_{CP}$$

or

$$z_{CP} = -\frac{1{,}518{,}000}{108{,}800} = -13.95 \text{ ft} \qquad \textit{Ans. (b)}$$

The center of pressure of the total resultant force on the right side of the tank lies 13.95 ft below the surface.

2.8 Buoyancy and Stability

The same principles used to compute hydrostatic forces on surfaces can be applied to the net pressure force on a completely submerged or floating body. The results are the two laws of buoyancy discovered by Archimedes in the third century B.C.:

1. A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces.
2. A floating body displaces its own weight in the fluid in which it floats.

These two laws are easily derived by referring to Fig. 2.16. In Fig. 2.16*a*, the body lies between an upper curved surface 1 and a lower curved surface 2. From Eq. (2.45) for vertical force, the body experiences a net upward force

$$\begin{aligned} F_B &= F_V(2) - F_V(1) \\ &= (\text{fluid weight above 2}) - (\text{fluid weight above 1}) \\ &= \text{weight of fluid equivalent to body volume} \end{aligned} \qquad (2.48)$$

Alternatively, from Fig. 2.16*b*, we can sum the vertical forces on elemental vertical slices through the immersed body:

$$F_B = \int_{\text{body}} (p_2 - p_1)\, dA_H = -\gamma \int (z_2 - z_1)\, dA_H = (\gamma)(\text{body volume}) \qquad (2.49)$$

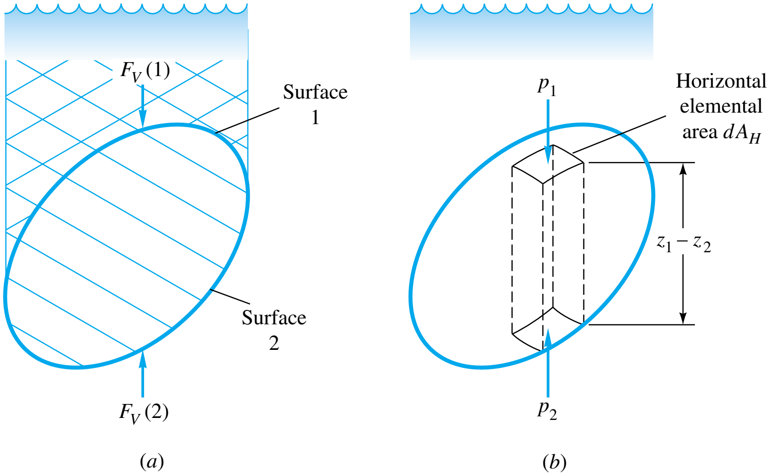


Fig. 2.16 Two different approaches to the buoyant force on an arbitrary immersed body: (*a*) forces on upper and lower curved surfaces; (*b*) summation of elemental vertical-pressure forces.

These are identical results and equivalent to law 1 above.

Equation (2.49) assumes that the fluid has uniform specific weight. The line of action of the buoyant force passes through the center of volume of the displaced body; i.e., its center of mass is computed as if it had uniform density. This point through which F_B acts is called the *center of buoyancy*, commonly labeled B or CB on a drawing. Of course, the point B may or may not correspond to the actual center of mass of the body's own material, which may have variable density.

Equation (2.49) can be generalized to a layered fluid (LF) by summing the weights of each layer of density ρ_i displaced by the immersed body:

$$(F_B)_{\text{LF}} = \sum \rho_i g(\text{displaced volume})_i \tag{2.50}$$

Each displaced layer would have its own center of volume, and one would have to sum moments of the incremental buoyant forces to find the center of buoyancy of the immersed body.

Since liquids are relatively heavy, we are conscious of their buoyant forces, but gases also exert buoyancy on any body immersed in them. For example, human beings have an average specific weight of about 60 lbf/ft^3. We may record the weight of a person as 180 lbf and thus estimate the person's total volume as 3.0 ft^3. However, in so doing we are neglecting the buoyant force of the air surrounding the person. At standard conditions, the specific weight of air is 0.0763 lbf/ft^3; hence the buoyant force is approximately 0.23 lbf. If measured in vacuo, the person would weigh about 0.23 lbf more. For balloons and blimps the buoyant force of air, instead of being negligible, is the controlling factor in the design. Also, many flow phenomena, e.g., natural convection of heat and vertical mixing in the ocean, are strongly dependent upon seemingly small buoyant forces.

Floating bodies are a special case; only a portion of the body is submerged, with the remainder poking up out of the free surface. This is illustrated in Fig. 2.17, where the shaded portion is the displaced volume. Equation (2.49) is modified to apply to this smaller volume

$$F_B = (\gamma)(\text{displaced volume}) = \text{floating-body weight} \tag{2.51}$$

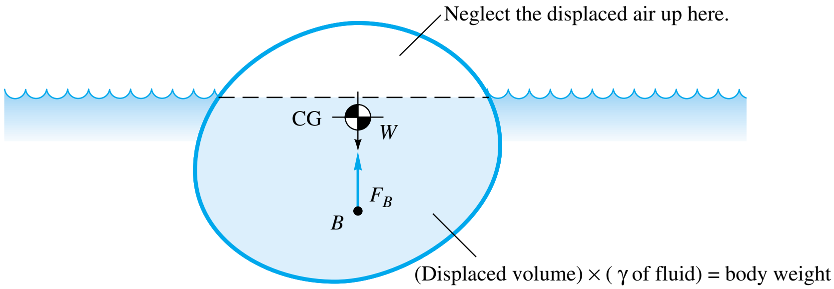
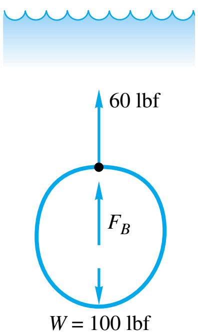


Fig. 2.17 Static equilibrium of a floating body.

Not only does the buoyant force equal the body weight, but also they are *collinear* since there can be no net moments for static equilibrium. Equation (2.51) is the mathematical equivalent of Archimedes' law 2, previously stated.

EXAMPLE 2.9

A block of concrete weighs 100 lbf in air and "weighs" only 60 lbf when immersed in fresh water (62.4 lbf/ft^3). What is the average specific weight of the block?



E2.9

Solution

A free-body diagram of the submerged block (see Fig. E2.9) shows a balance between the apparent weight, the buoyant force, and the actual weight

$$\sum F_z = 0 = 60 + F_B - 100$$

or

$$F_B = 40 \text{ lbf} = (62.4 \text{ lbf/ft}^3)(\text{block volume, ft}^3)$$

Solving gives the volume of the block as 40/62.4 = 0.641 ft^3. Therefore the specific weight of the block is

$$\gamma_{\text{block}} = \frac{100 \text{ lbf}}{0.641 \text{ ft}^3} = 156 \text{ lbf/ft}^3 \qquad \textit{Ans.}$$

Occasionally, a body will have exactly the right weight and volume for its ratio to equal the specific weight of the fluid. If so, the body will be *neutrally buoyant* and will remain at rest at any point where it is immersed in the fluid. Small neutrally buoyant particles are sometimes used in flow visualization, and a neutrally buoyant body called a *Swallow float* [2] is used to track oceanographic currents. A submarine can achieve positive, neutral, or negative buoyancy by pumping water in or out of its ballast tanks.

Stability

A floating body as in Fig. 2.17 may not approve of the position in which it is floating. If so, it will overturn at the first opportunity and is said to be statically *unstable*, like a pencil balanced upon its point. The least disturbance will cause it to seek another equilibrium position which is stable. Engineers must design to avoid floating instabil-

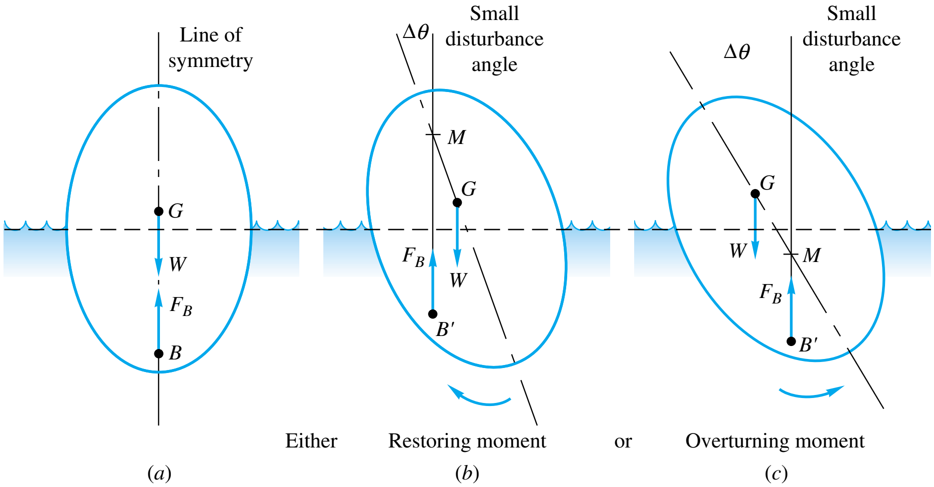


Fig. 2.18 Calculation of the metacenter M of the floating body shown in (*a*). Tilt the body a small angle $\Delta\theta$. Either (*b*) B' moves far out (point M above G denotes stability); or (*c*) B' moves slightly (point M below G denotes instability).

ity. The only way to tell for sure whether a floating position is stable is to "disturb" the body a slight amount mathematically and see whether it develops a restoring moment which will return it to its original position. If so, it is stable; if not, unstable. Such calculations for arbitrary floating bodies have been honed to a fine art by naval architects [3], but we can at least outline the basic principle of the static-stability calculation. Figure 2.18 illustrates the computation for the usual case of a symmetric floating body. The steps are as follows:

1. The basic floating position is calculated from Eq. (2.51). The body's center of mass G and center of buoyancy B are computed.
2. The body is tilted a small angle $\Delta\theta$, and a new waterline is established for the body to float at this angle. The new position B' of the center of buoyancy is calculated. A vertical line drawn upward from B' intersects the line of symmetry at a point M, called the *metacenter*, which is independent of $\Delta\theta$ for small angles.
3. If point M is above G, that is, if the *metacentric height* $\overline{MG}$ is positive, a restoring moment is present and the original position is stable. If M is below G (negative $\overline{MG}$, the body is unstable and will overturn if disturbed. Stability increases with increasing $\overline{MG}$.

Thus the metacentric height is a property of the cross section for the given weight, and its value gives an indication of the stability of the body. For a body of varying cross section and draft, such as a ship, the computation of the metacenter can be very involved.

Stability Related to Waterline Area

Naval architects [3] have developed the general stability concepts from Fig. 2.18 into a simple computation involving the area moment of inertia of the *waterline area* about the axis of tilt. The derivation assumes that the body has a smooth shape variation (no discontinuities) near the waterline and is derived from Fig. 2.19.

The y-axis of the body is assumed to be a line of symmetry. Tilting the body a small angle θ then submerges small wedge Obd and uncovers an equal wedge cOa, as shown.

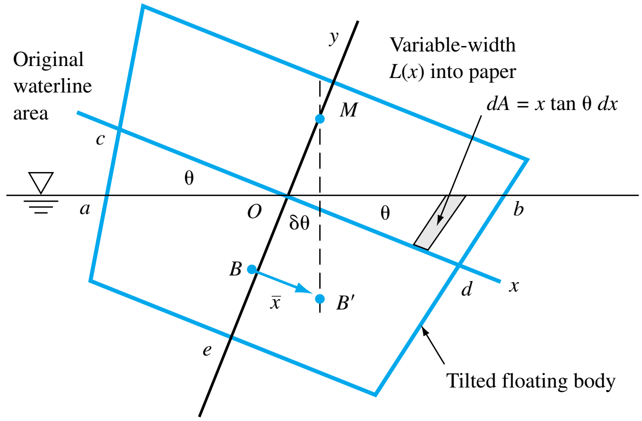


Fig. 2.19 A floating body tilted through a small angle θ. The movement $\bar{x}$ of the center of buoyancy B is related to the waterline area moment of inertia.

The new position B' of the center of buoyancy is calculated as the centroid of the submerged portion *aObde* of the body:

$$\bar{x}\,\upsilon_{abOde} = \int_{cOdea} x\, d\upsilon + \int_{Obd} x\, d\upsilon - \int_{cOa} x\, d\upsilon = 0 + \int_{Obd} x\,(L\, dA) - \int_{cOa} x\,(L\, dA)$$

$$= 0 + \int_{Obd} x\, L\,(x \tan\theta\, dx) - \int_{cOa} xL\,(-x \tan\theta\, dx) = \tan\theta \int_{\text{waterline}} x^2\, dA_{\text{waterline}} = I_O \tan\theta$$

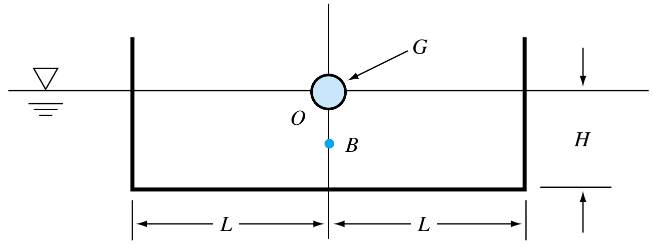
where I_O is the area moment of inertia of the *waterline footprint* of the body about its tilt axis O. The first integral vanishes because of the symmetry of the original submerged portion *cOdea*. The remaining two "wedge" integrals combine into I_O when we notice that $L\,dx$ equals an element of *waterline area*. Thus we determine the desired distance from M to B:

$$\frac{\bar{x}}{\quad} = \overline{MB} = \frac{I_O}{\upsilon_{\text{submerged}}} = \overline{MG} + \overline{GB} \qquad \text{or} \qquad \overline{MG} = \frac{I_O}{\upsilon_{\text{sub}}} - \overline{GB} \tag{2.52}$$

The engineer would determine the distance from G to B from the basic shape and design of the floating body and then make the calculation of I_O and the submerged volume υ_{sub}. If the metacentric height MG is positive, the body is stable for small disturbances. Note that if $\overline{GB}$ is negative, that is, B is *above* G, the body is always stable.

EXAMPLE 2.10

A barge has a uniform rectangular cross section of width $2L$ and vertical draft of height H, as in Fig. E2.10. Determine (*a*) the metacentric height for a small tilt angle and (*b*) the range of ratio L/H for which the barge is statically stable if G is exactly at the waterline as shown.



E2.10

Solution

If the barge has length b into the paper, the waterline area, relative to tilt axis O, has a base b and a height $2L$; therefore, $I_O = b(2L)^3/12$. Meanwhile, $\upsilon_{sub} = 2LbH$. Equation (2.52) predicts

$$\overline{MG} = \frac{I_O}{\upsilon_{sub}} - \overline{GB} = \frac{8bL^3/12}{2LbH} - \frac{H}{2} = \frac{L^2}{3H} - \frac{H}{2} \qquad \textit{Ans. (a)}$$

The barge can thus be stable only if

$$L^2 > 3H^2/2 \qquad \text{or} \qquad 2L > 2.45H \qquad \textit{Ans. (b)}$$

The wider the barge relative to its draft, the more stable it is. Lowering G would help also.

Even an expert will have difficulty determining the floating stability of a buoyant body of irregular shape. Such bodies may have two or more stable positions. For example, a ship may float the way we like it, so that we can sit upon the deck, or it may float upside down (capsized). An interesting mathematical approach to floating stability is given in Ref. 11. The author of this reference points out that even simple shapes, e.g., a cube of uniform density, may have a great many stable floating orientations, not necessarily symmetric. Homogeneous circular cylinders can float with the axis of symmetry tilted from the vertical.

Floating instability occurs in nature. Living fish generally swim with their plane of symmetry vertical. After death, this position is unstable and they float with their flat sides up. Giant icebergs may overturn after becoming unstable when their shapes change due to underwater melting. Iceberg overturning is a dramatic, rarely seen event.

Figure 2.20 shows a typical North Atlantic iceberg formed by calving from a Greenland glacier which protruded into the ocean. The exposed surface is rough, indicating that it has undergone further calving. Icebergs are frozen fresh, bubbly, glacial water of average density 900 kg/m^3. Thus, when an iceberg is floating in seawater, whose average density is 1025 kg/m^3, approximately 900/1025, or seven-eighths, of its volume lies below the water.

2.9 Pressure Distribution in Rigid-Body Motion

In rigid-body motion, all particles are in combined translation and rotation, and there is no relative motion between particles. With no relative motion, there are no strains

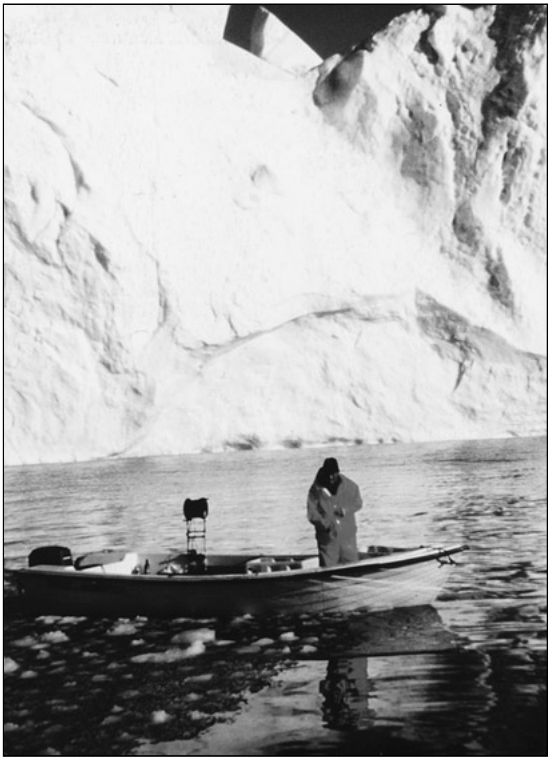


Fig. 2.20 A North Atlantic iceberg formed by calving from a Greenland glacier. These, and their even larger Antarctic sisters, are the largest floating bodies in the world. Note the evidence of further calving fractures on the front surface. *(Courtesy of Søren Thalund, Greenland tourism a/s Iiulissat, Greenland.)*

or strain rates, so that the viscous term $\mu\nabla^2\mathbf{V}$ in Eq. (2.13) vanishes, leaving a balance between pressure, gravity, and particle acceleration

$$\nabla p = \rho(\mathbf{g} - \mathbf{a}) \tag{2.53}$$

The pressure gradient acts in the direction $\mathbf{g} - \mathbf{a}$, and lines of constant pressure (including the free surface, if any) are perpendicular to this direction. The general case of combined translation and rotation of a rigid body is discussed in Chap. 3, Fig. 3.12. If the center of rotation is at point O and the translational velocity is V_0 at this point, the velocity of an arbitrary point P on the body is given by[2]

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{\Omega} \times \mathbf{r}_0$$

where $\mathbf{\Omega}$ is the angular-velocity vector and $\mathbf{r}_0$ is the position of point P. Differentiating, we obtain the most general form of the acceleration of a rigid body:

$$\mathbf{a} = \frac{d\mathbf{V}_0}{dt} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_0) + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r}_0 \tag{2.54}$$

Looking at the right-hand side, we see that the first term is the translational acceleration; the second term is the *centripetal acceleration*, whose direction is from point

[2] For a more detailed derivation of rigid-body motion, see Ref. 4, Sec. 2.7.

P perpendicular toward the axis of rotation; and the third term is the linear acceleration due to changes in the angular velocity. It is rare for all three of these terms to apply to any one fluid flow. In fact, fluids can rarely move in rigid-body motion unless restrained by confining walls for a long time. For example, suppose a tank of water is in a car which starts a constant acceleration. The water in the tank would begin to slosh about, and that sloshing would damp out very slowly until finally the particles of water would be in approximately rigid-body acceleration. This would take so long that the car would have reached hypersonic speeds. Nevertheless, we can at least discuss the pressure distribution in a tank of rigidly accelerating water. The following is an example where the water in the tank will reach uniform acceleration rapidly.

EXAMPLE 2.11

A tank of water 1 m deep is in free fall under gravity with negligible drag. Compute the pressure at the bottom of the tank if $p_a = 101$ kPa.

Solution

Being unsupported in this condition, the water particles tend to fall downward as a rigid hunk of fluid. In free fall with no drag, the downward acceleration is $\mathbf{a} = \mathbf{g}$. Thus Eq. (2.53) for this situation gives $\nabla p = \rho(\mathbf{g} - \mathbf{g}) = 0$. The pressure in the water is thus *constant everywhere* and equal to the atmospheric pressure 101 kPa. In other words, the walls are doing no service in sustaining the pressure distribution which would normally exist.

Uniform Linear Acceleration

In this general case of uniform rigid-body acceleration, Eq. (2.53) applies, **a** having the same magnitude and direction for all particles. With reference to Fig. 2.21, the parallelogram sum of **g** and **−a** gives the direction of the pressure gradient or greatest rate of increase of p. The surfaces of constant pressure must be perpendicular to this and are thus tilted at a downward angle θ such that

$$\theta = \tan^{-1}\frac{a_x}{g + a_z} \qquad (2.55)$$

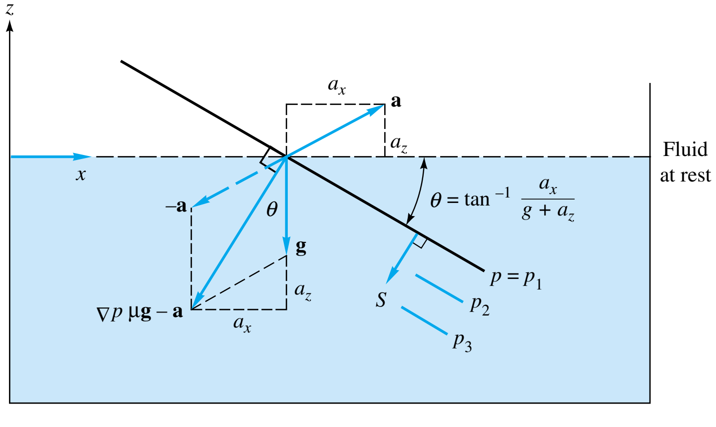


Fig. 2.21 Tilting of constant-pressure surfaces in a tank of liquid in rigid-body acceleration.

One of these tilted lines is the free surface, which is found by the requirement that the fluid retain its volume unless it spills out. The rate of increase of pressure in the direction **g** − **a** is greater than in ordinary hydrostatics and is given by

$$\frac{dp}{ds} = \rho G \qquad \text{where } G = [a_x^2 + (g + a_z)^2]^{1/2} \tag{2.56}$$

These results are independent of the size or shape of the container as long as the fluid is continuously connected throughout the container.

EXAMPLE 2.12

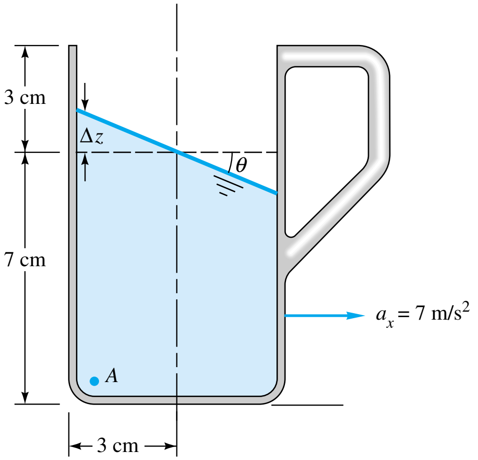
A drag racer rests her coffee mug on a horizontal tray while she accelerates at 7 m/s^2. The mug is 10 cm deep and 6 cm in diameter and contains coffee 7 cm deep at rest. (*a*) Assuming rigid-body acceleration of the coffee, determine whether it will spill out of the mug. (*b*) Calculate the gage pressure in the corner at point *A* if the density of coffee is 1010 kg/m^3.

Solution

Part (a) The free surface tilts at the angle θ given by Eq. (2.55) regardless of the shape of the mug. With $a_z = 0$ and standard gravity,

$$\theta = \tan^{-1}\frac{a_x}{g} = \tan^{-1}\frac{7.0}{9.81} = 35.5°$$

If the mug is symmetric about its central axis, the volume of coffee is conserved if the tilted surface intersects the original rest surface exactly at the centerline, as shown in Fig. E2.12.



E2.12

Thus the deflection at the left side of the mug is

$$z = (3\text{ cm})(\tan\theta) = 2.14\text{ cm} \qquad \textit{Ans. (a)}$$

This is less than the 3-cm clearance available, so the coffee will not spill unless it was sloshed during the start-up of acceleration.

Part (b) When at rest, the gage pressure at point *A* is given by Eq. (2.20):

$$p_A = \rho g(z_{surf} - z_A) = (1010\text{ kg/m}^3)(9.81\text{ m/s}^2)(0.07\text{ m}) = 694\text{ N/m}^2 = 694\text{ Pa}$$

During acceleration, Eq. (2.56) applies, with $G = [(7.0)^2 + (9.81)^2]^{1/2} = 12.05 \text{ m/s}^2$. The distance Δs down the normal from the tilted surface to point A is

$$\Delta s = (7.0 + 2.14)(\cos \theta) = 7.44 \text{ cm}$$

Thus the pressure at point A becomes

$$p_A = \rho G \, \Delta s = 1010(12.05)(0.0744) = 906 \text{ Pa} \qquad \textit{Ans. (b)}$$

which is an increase of 31 percent over the pressure when at rest.

Rigid-Body Rotation

As a second special case, consider rotation of the fluid about the z axis without any translation, as sketched in Fig. 2.22. We assume that the container has been rotating long enough at constant Ω for the fluid to have attained rigid-body rotation. The fluid acceleration will then be the centripetal term in Eq. (2.54). In the coordinates of Fig. 2.22, the angular-velocity and position vectors are given by

$$\boldsymbol{\Omega} = \mathbf{k}\Omega \qquad r_0 = \mathbf{i}_r r \tag{2.57}$$

Then the acceleration is given by

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_0) = -r\Omega^2 \mathbf{i}_r \tag{2.58}$$

as marked in the figure, and Eq. (2.53) for the force balance becomes

$$\nabla p = \mathbf{i}_r \frac{\partial p}{\partial r} + \mathbf{k} \frac{\partial p}{\partial z} = \rho(\mathbf{g} - \mathbf{a}) = \rho(-g\mathbf{k} + r\Omega^2 \mathbf{i}_r) \tag{2.59}$$

Equating like components, we find the pressure field by solving two first-order partial differential equations

$$\frac{\partial p}{\partial r} = \rho r \Omega^2 \qquad \frac{\partial p}{\partial z} = -\gamma \tag{2.60}$$

This is our first specific example of the generalized three-dimensional problem described by Eqs. (2.14) for more than one independent variable. The right-hand sides of

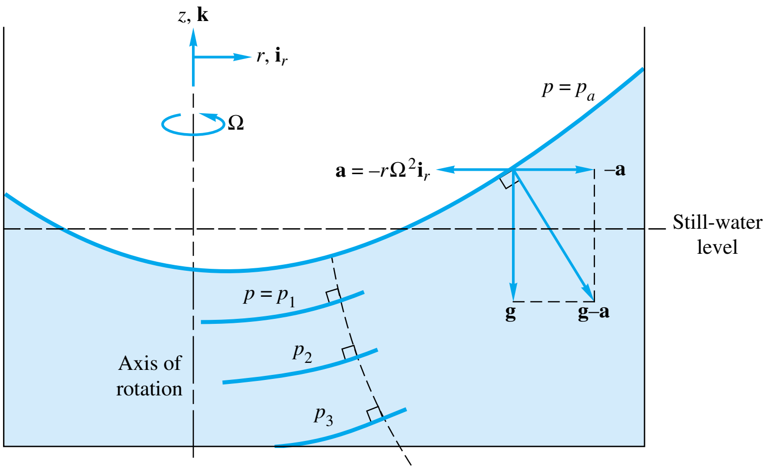


Fig. 2.22 Development of paraboloid constant-pressure surfaces in a fluid in rigid-body rotation. The dashed line along the direction of maximum pressure increase is an exponential curve.

(2.60) are known functions of r and z. One can proceed as follows: Integrate the first equation "partially," i.e., holding z constant, with respect to r. The result is

$$p = \tfrac{1}{2}\rho r^2\Omega^2 + f(z) \tag{2.61}$$

where the "constant" of integration is actually a function $f(z)$.† Now differentiate this with respect to z and compare with the second relation of (2.60):

$$\frac{\partial p}{\partial z} = 0 + f'(z) = -\gamma$$

or

$$f(z) = -\gamma z + C \tag{2.62a}$$

where C is a constant. Thus Eq. (2.61) now becomes

$$p = \text{const} - \gamma z + \tfrac{1}{2}\rho r^2\Omega^2 \tag{2.62b}$$

This is the pressure distribution in the fluid. The value of C is found by specifying the pressure at one point. If $p = p_0$ at $(r, z) = (0, 0)$, then $C = p_0$. The final desired distribution is

$$p = p_0 - \gamma_z + \tfrac{1}{2}\rho r^2\Omega^2 \tag{2.63}$$

The pressure is linear in z and parabolic in r. If we wish to plot a constant-pressure surface, say, $p = p_1$, Eq. (2.63) becomes

$$z = \frac{p_0 - p_1}{\gamma} + \frac{r^2\Omega^2}{2g} = a + br^2 \tag{2.64}$$

Thus the surfaces are paraboloids of revolution, concave upward, with their minimum point on the axis of rotation. Some examples are sketched in Fig. 2.22.

As in the previous example of linear acceleration, the position of the free surface is found by conserving the volume of fluid. For a noncircular container with the axis of rotation off-center, as in Fig. 2.22, a lot of laborious mensuration is required, and a single problem will take you all weekend. However, the calculation is easy for a cylinder rotating about its central axis, as in Fig. 2.23. Since the volume of a paraboloid is

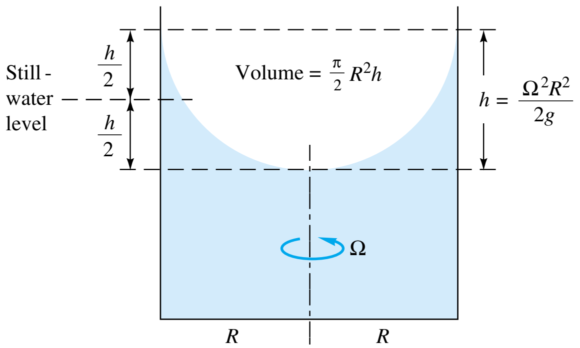


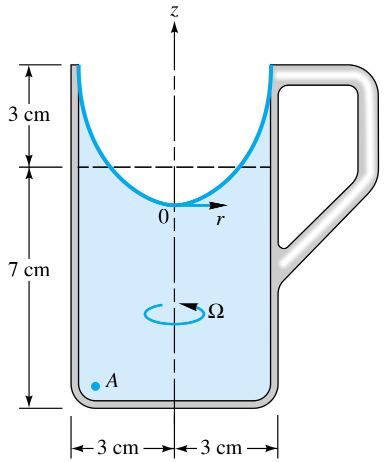
Fig. 2.23 Determining the free-surface position for rotation of a cylinder of fluid about its central axis.

†This is because $f(z)$ vanishes when differentiated with respect to r. If you don't see this, you should review your calculus.

one-half the base area times its height, the still-water level is exactly halfway between the high and low points of the free surface. The center of the fluid drops an amount $h/2 = \Omega^2 R^2/(4g)$, and the edges rise an equal amount.

EXAMPLE 2.13

The coffee cup in Example 2.12 is removed from the drag racer, placed on a turntable, and rotated about its central axis until a rigid-body mode occurs. Find (*a*) the angular velocity which will cause the coffee to just reach the lip of the cup and (*b*) the gage pressure at point *A* for this condition.



E2.13

Solution

Part (a) The cup contains 7 cm of coffee. The remaining distance of 3 cm up to the lip must equal the distance $h/2$ in Fig. 2.23. Thus

$$\frac{h}{2} = 0.03\ \text{m} = \frac{\Omega^2 R^2}{4g} = \frac{\Omega^2(0.03\ \text{m})^2}{4(9.81\ \text{m/s}^2)}$$

Solving, we obtain

$$\Omega^2 = 1308 \qquad \text{or} \qquad \Omega = 36.2\ \text{rad/s} = 345\ \text{r/min} \qquad \textit{Ans. (a)}$$

Part (b) To compute the pressure, it is convenient to put the origin of coordinates *r* and *z* at the bottom of the free-surface depression, as shown in Fig. E2.13. The gage pressure here is $p_0 = 0$, and point *A* is at $(r, z) = (3\ \text{cm}, -4\ \text{cm})$. Equation (2.63) can then be evaluated

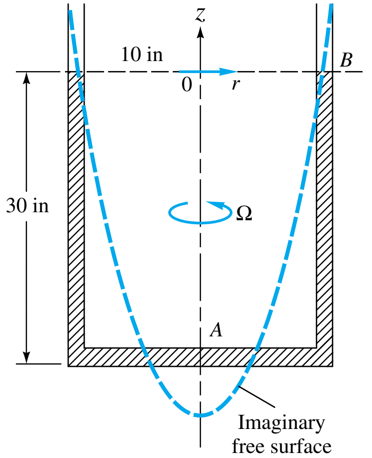
$$p_A = 0 - (1010\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(-0.04\ \text{m})$$
$$+ \tfrac{1}{2}(1010\ \text{kg/m}^3)(0.03\ \text{m})^2(1308\ \text{rad}^2/\text{s}^2)$$
$$= 396\ \text{N/m}^2 + 594\ \text{N/m}^2 = 990\ \text{Pa} \qquad \textit{Ans. (b)}$$

This is about 43 percent greater than the still-water pressure $p_A = 694$ Pa.

Here, as in the linear-acceleration case, it should be emphasized that the paraboloid pressure distribution (2.63) sets up in *any* fluid under rigid-body rotation, regardless of the shape or size of the container. The container may even be closed and filled with fluid. It is only necessary that the fluid be continuously interconnected throughout the container. The following example will illustrate a peculiar case in which one can visualize an imaginary free surface extending outside the walls of the container.

EXAMPLE 2.14

A U-tube with a radius of 10 in and containing mercury to a height of 30 in is rotated about its center at 180 r/min until a rigid-body mode is achieved. The diameter of the tubing is negligible. Atmospheric pressure is 2116 lbf/ft^2. Find the pressure at point *A* in the rotating condition. See Fig. E2.14.



E2.14

Solution

Convert the angular velocity to radians per second:

$$\Omega = (180 \text{ r/min}) \frac{2\pi \text{ rad/r}}{60 \text{ s/min}} = 18.85 \text{ rad/s}$$

From Table 2.1 we find for mercury that $\gamma = 846$ lbf/ft^3 and hence $\rho = 846/32.2 = 26.3$ slugs/ft^3. At this high rotation rate, the free surface will slant upward at a fierce angle [about 84°; check this from Eq. (2.64)], but the tubing is so thin that the free surface will remain at approximately the same 30-in height, point B. Placing our origin of coordinates at this height, we can calculate the constant C in Eq. (2.62b) from the condition $p_B = 2116$ lbf/ft^2 at $(r, z) = (10 \text{ in}, 0)$:

$$p_B = 2116 \text{ lbf/ft}^2 = C - 0 + \tfrac{1}{2}(26.3 \text{ slugs/ft}^3)(\tfrac{10}{12} \text{ ft})^2(18.85 \text{ rad/s})^2$$

or

$$C = 2116 - 3245 = -1129 \text{ lbf/ft}^2$$

We then obtain p_A by evaluating Eq. (2.63) at $(r, z) = (0, -30 \text{ in})$:

$$p_A = -1129 - (846 \text{ lbf/ft}^3)(-\tfrac{30}{12} \text{ ft}) = -1129 + 2115 = 986 \text{ lbf/ft}^2 \qquad \textit{Ans.}$$

This is less than atmospheric pressure, and we can see why if we follow the free-surface paraboloid down from point B along the dashed line in the figure. It will cross the horizontal portion of the U-tube (where p will be atmospheric) and fall *below* point A. From Fig. 2.23 the actual drop from point B will be

$$h = \frac{\Omega^2 R^2}{2g} = \frac{(18.85)^2(\frac{10}{12})^2}{2(32.2)} = 3.83 \text{ ft} = 46 \text{ in}$$

Thus p_A is about 16 inHg below atmospheric pressure, or about $\frac{16}{12}(846) = 1128$ lbf/ft^2 below $p_a = 2116$ lbf/ft^2, which checks with the answer above. When the tube is at rest,

$$p_A = 2116 - 846(-\tfrac{30}{12}) = 4231 \text{ lbf/ft}^2$$

Hence rotation has reduced the pressure at point A by 77 percent. Further rotation can reduce p_A to near-zero pressure, and cavitation can occur.

An interesting by-product of this analysis for rigid-body rotation is that the lines everywhere parallel to the pressure gradient form a family of curved surfaces, as sketched in Fig. 2.22. They are everywhere orthogonal to the constant-pressure surfaces, and hence their slope is the negative inverse of the slope computed from Eq. (2.64):

$$\left.\frac{dz}{dr}\right|_{\text{GL}} = -\frac{1}{(dz/dr)_{p=\text{const}}} = -\frac{1}{r\Omega^2/g}$$

where GL stands for gradient line

or

$$\frac{dz}{dr} = -\frac{g}{r\Omega^2} \qquad (2.65)$$

Separating the variables and integrating, we find the equation of the pressure-gradient surfaces

$$r = C_1 \exp\left(-\frac{\Omega^2 z}{g}\right) \tag{2.66}$$

Notice that this result and Eq. (2.64) are independent of the density of the fluid. In the absence of friction and Coriolis effects, Eq. (2.66) defines the lines along which the apparent net gravitational field would act on a particle. Depending upon its density, a small particle or bubble would tend to rise or fall in the fluid along these exponential lines, as demonstrated experimentally in Ref. 5. Also, buoyant streamers would align themselves with these exponential lines, thus avoiding any stress other than pure tension. Figure 2.24 shows the configuration of such streamers before and during rotation.

2.10 Pressure Measurement

Pressure is a derived property. It is the force per unit area as related to fluid molecular bombardment of a surface. Thus most pressure instruments only *infer* the pressure by calibration with a primary device such as a deadweight piston tester. There are many such instruments, both for a static fluid and a moving stream. The instrumentation texts in Refs. 7 to 10, 12, and 13 list over 20 designs for pressure measurement instruments. These instruments may be grouped into four categories:

1. *Gravity-based:* barometer, manometer, deadweight piston.
2. *Elastic deformation:* bourdon tube (metal and quartz), diaphragm, bellows, strain-gage, optical beam displacement.
3. *Gas behavior:* gas compression (McLeod gage), thermal conductance (Pirani gage), molecular impact (Knudsen gage), ionization, thermal conductivity, air piston.
4. *Electric output:* resistance (Bridgman wire gage), diffused strain gage, capacitative, piezoelectric, magnetic inductance, magnetic reluctance, linear variable differential transformer (LVDT), resonant frequency.

The gas-behavior gages are mostly special-purpose instruments used for certain scientific experiments. The deadweight tester is the instrument used most often for calibrations; for example, it is used by the U.S. National Institute for Standards and Technology (NIST). The barometer is described in Fig. 2.6.

The manometer, analyzed in Sec. 2.4, is a simple and inexpensive hydrostatic-principle device with no moving parts except the liquid column itself. Manometer measurements must not disturb the flow. The best way to do this is to take the measurement through a *static hole* in the wall of the flow, as illustrated for the two instruments in Fig. 2.25. The hole should be normal to the wall, and burrs should be avoided. If the hole is small enough (typically 1-mm diameter), there will be no flow into the measuring tube once the pressure has adjusted to a steady value. Thus the flow is almost undisturbed. An oscillating flow pressure, however, can cause a large error due to possible dynamic response of the tubing. Other devices of smaller dimensions are used for dynamic-pressure measurements. Note that the manometers in Fig. 2.25 are arranged to measure the absolute pressures p_1 and p_2. If the pressure difference $p_1 - p_2$ is de-

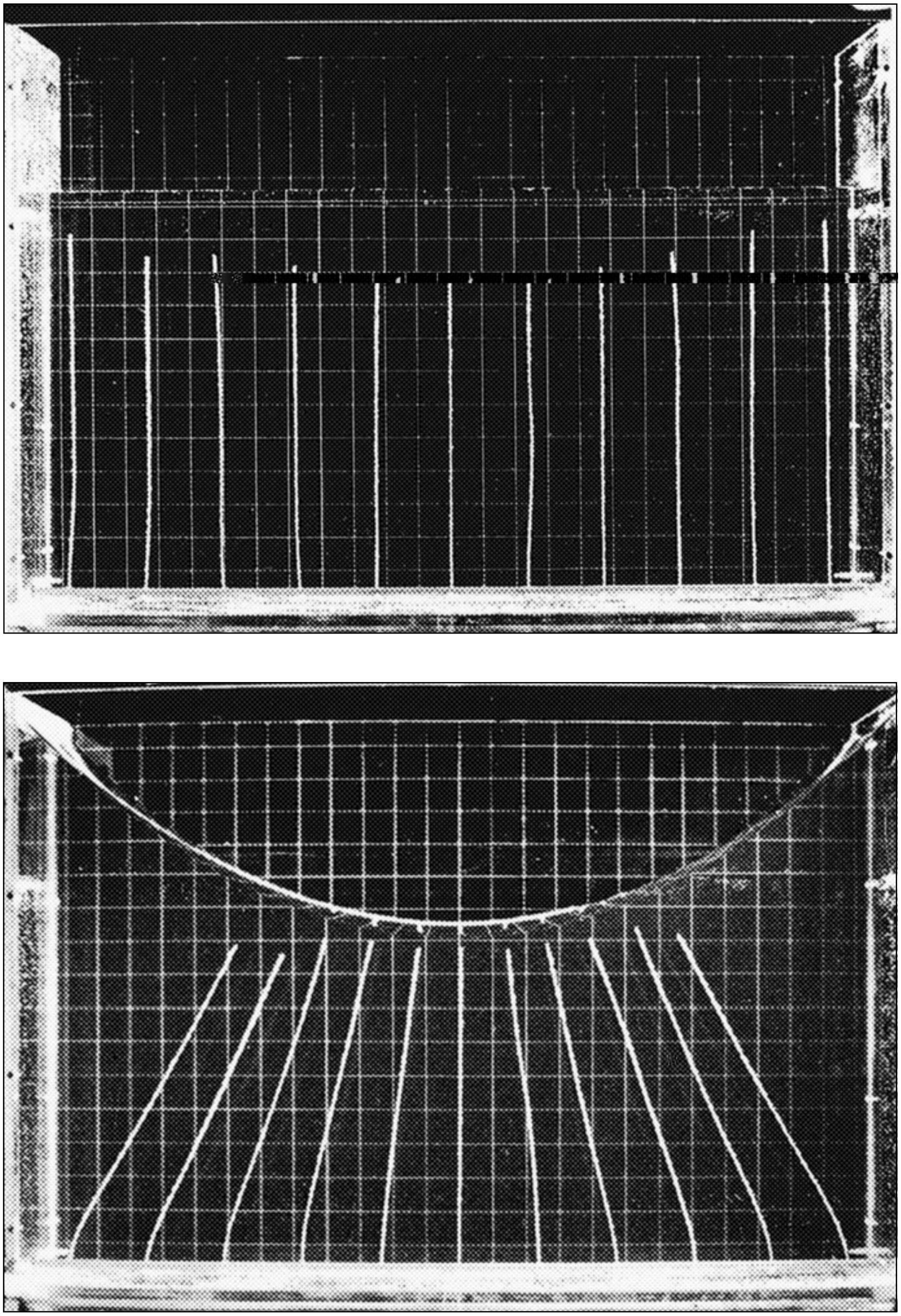


Fig. 2.24 Experimental demonstration with buoyant streamers of the fluid force field in rigid-body rotation: (*top*) fluid at rest (streamers hang vertically upward); (*bottom*) rigid-body rotation (streamers are aligned with the direction of maximum pressure gradient). (*From Ref. 5, courtesy of R. Ian Fletcher.*)

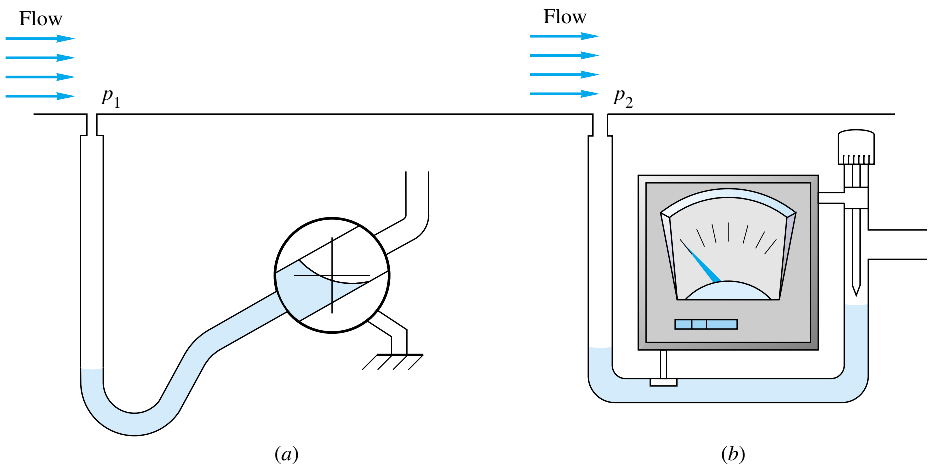


Fig. 2.25 Two types of accurate manometers for precise measurements: (*a*) tilted tube with eyepiece; (*b*) micrometer pointer with ammeter detector.

sired, a significant error is incurred by subtracting two independent measurements, and it would be far better to connect both ends of one instrument to the two static holes p_1 and p_2 so that one manometer reads the difference directly. In category 2, elastic-deformation instruments, a popular, inexpensive, and reliable device is the *bourdon tube,* sketched in Fig. 2.26. When pressurized internally, a curved tube with flattened cross section will deflect outward. The deflection can be measured by a linkage attached to a calibrated dial pointer, as shown. Or the deflection can be used to drive electric-output sensors, such as a variable transformer. Similarly, a membrane or *diaphragm* will deflect under pressure and can either be sensed directly or used to drive another sensor.

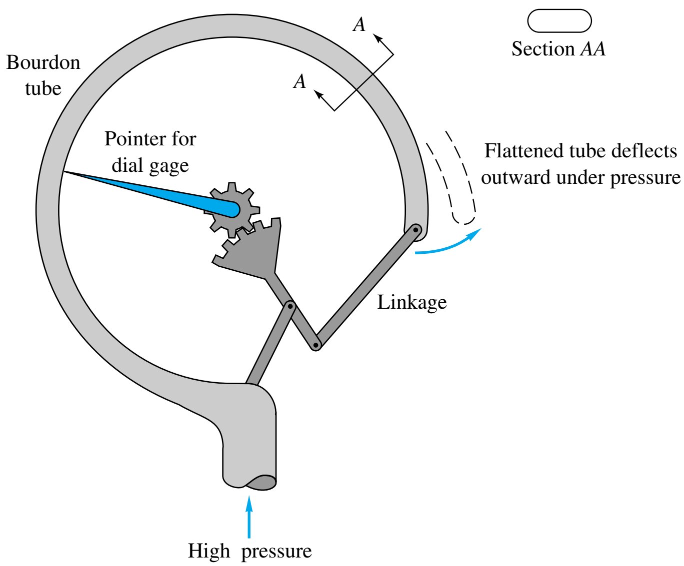


Fig. 2.26 Schematic of a bourdon-tube device for mechanical measurement of high pressures.

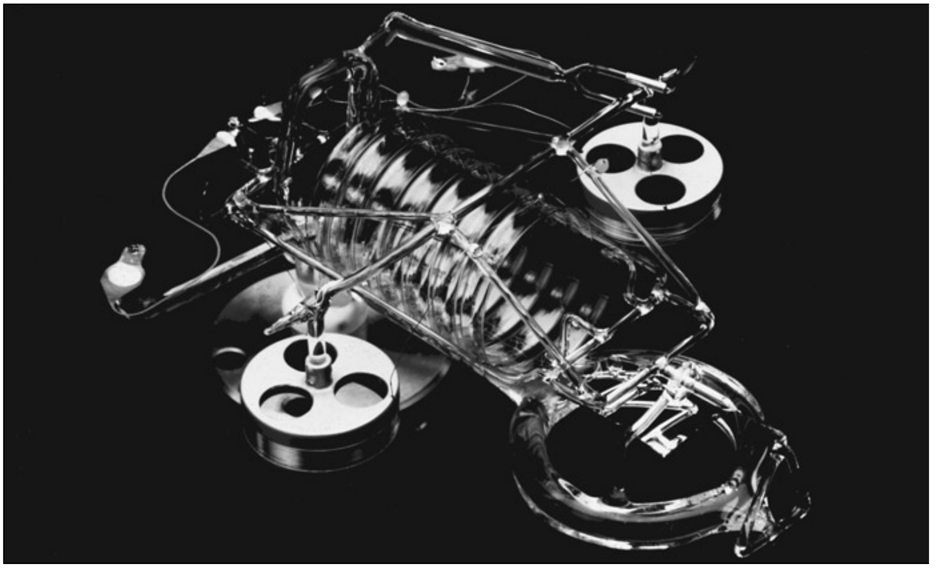


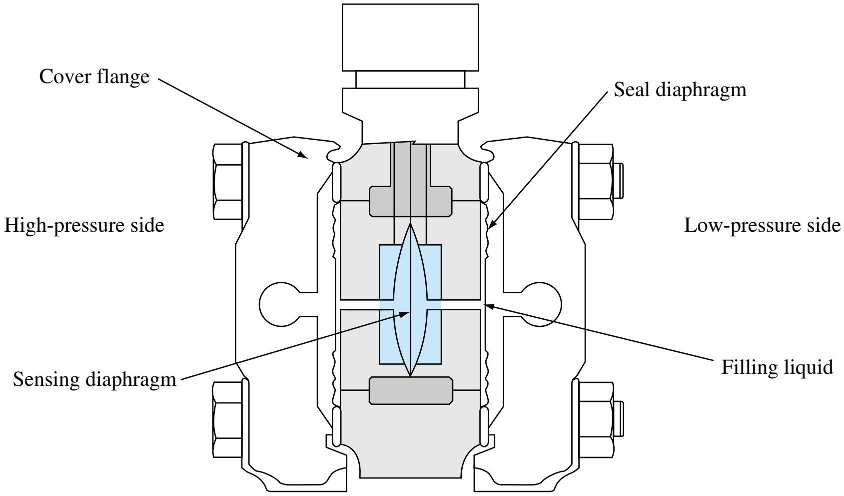
Fig. 2.27 The fused-quartz, force-balanced bourdon tube is the most accurate pressure sensor used in commercial applications today. (*Courtesy of Ruska Instrument Corporation, Houston, TX.*)

An interesting variation of Fig. 2.26 is the *fused-quartz, forced-balanced bourdon tube,* shown in Fig. 2.27, whose deflection is sensed optically and returned to a zero reference state by a magnetic element whose output is proportional to the fluid pressure. The fused-quartz, forced-balanced bourdon tube is reported to be one of the most accurate pressure sensors ever devised, with uncertainty of the order of ± 0.003 percent.

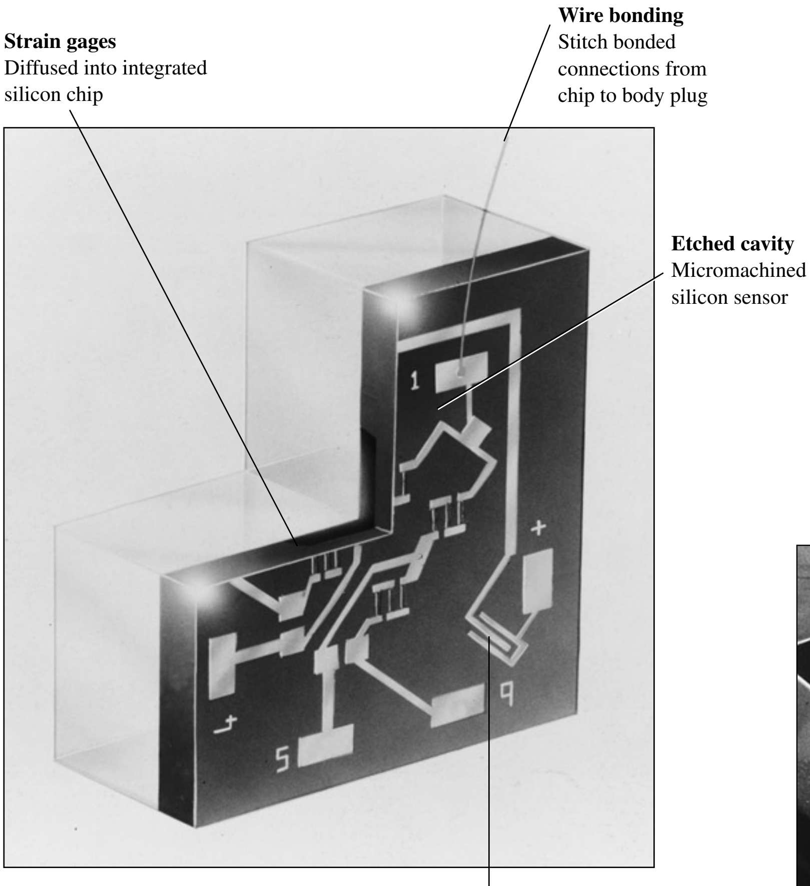
The last category, *electric-output* sensors, is extremely important in engineering because the data can be stored on computers and freely manipulated, plotted, and analyzed. Three examples are shown in Fig. 2.28, the first being the *capacitive* sensor in Fig. 2.28*a*. The differential pressure deflects the silicon diaphragm and changes the capacitancce of the liquid in the cavity. Note that the cavity has spherical end caps to prevent overpressure damage. In the second type, Fig. 2.28*b*, strain gages and other sensors are diffused or etched onto a chip which is stressed by the applied pressure. Finally, in Fig. 2.28*c*, a micromachined silicon sensor is arranged to deform under pressure such that its natural vibration frequency is proportional to the pressure. An oscillator excites the element's resonant frequency and converts it into appropriate pressure units. For further information on pressure sensors, see Refs. 7 to 10, 12, and 13.

Summary

This chapter has been devoted entirely to the computation of pressure distributions and the resulting forces and moments in a static fluid or a fluid with a known velocity field. All hydrostatic (Secs. 2.3 to 2.8) and rigid-body (Sec. 2.9) problems are solved in this manner and are classic cases which every student should understand. In arbitrary viscous flows, both pressure and velocity are unknowns and are solved together as a system of equations in the chapters which follow.

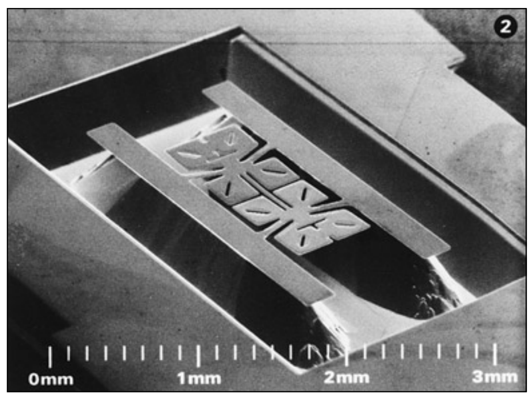


(*a*)



(*b*)

Fig. 2.28 Pressure sensors with electric output: (*a*) a silicon diaphragm whose deflection changes the cavity capacitance (*Courtesy of Johnson-Yokogawa Inc.*); (*b*) a silicon strain gage which is stressed by applied pressure; (*c*) a micromachined silicon element which resonates at a frequency proportional to applied pressure. *[(b) and (c) are courtesy of Druck, Inc., Fairfield, CT.]*



(*c*)