In this problem we assumed that the velocity is uniform across the section. In fact, the velocity in the bend approximates a free vortex (irrotational) profile in which $V \propto 1/r$ (where r is the radius) instead of $V = \text{const}$. Hence, this flow-measurement device could only be used to obtain approximate values of the flow rate.

6-3 BERNOULLI EQUATION—INTEGRATION OF EULER'S EQUATION ALONG A STREAMLINE FOR STEADY FLOW

Compared to the viscous-flow equivalents, the momentum or Euler's equation for incompressible, inviscid flow (Eqs. 6.1) is simpler mathematically, but solution (in conjunction with the mass conservation equation, Eq. 5.1c) still presents formidable difficulties in all but the most basic flow problems. One convenient approach for a steady flow is to integrate Euler's equation along a streamline. We will do this below using two different mathematical approaches, and each will result in the Bernoulli equation. Recall that in Section 4-4 we derived the Bernoulli equation by starting with a differential control volume; these two additional derivations will give us more insight into the restrictions inherent in use of the Bernoulli equation.

Derivation Using Streamline Coordinates

Euler's equation for steady flow along a streamline (from Eq. 6.4a) is

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s} \tag{6.6}$$

If a fluid particle moves a distance, ds, along a streamline, then

$$\frac{\partial p}{\partial s}\, ds = dp \qquad \text{(the change in pressure along } s\text{)}$$

$$\frac{\partial z}{\partial s}\, ds = dz \qquad \text{(the change in elevation along } s\text{)}$$

$$\frac{\partial V}{\partial s}\, ds = dV \qquad \text{(the change in speed along } s\text{)}$$

Thus, after multiplying Eq. 6.6 by ds, we can write

$$-\frac{dp}{\rho} - g\,dz = V\,dV \qquad \text{or} \qquad \frac{dp}{\rho} + V\,dV + g\,dz = 0 \qquad \text{(along } s\text{)}$$

Integration of this equation gives

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \qquad \text{(along } s\text{)} \tag{6.7}$$

Before Eq. 6.7 can be applied, we must specify the relation between pressure and density. For the special case of incompressible flow, $\rho = \text{constant}$, and Eq. 6.7 becomes the Bernoulli equation,

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{6.8}$$

Restrictions: (1) Steady flow.
(2) Incompressible flow.
(3) Frictionless flow.
(4) Flow along a streamline.

The Bernoulli equation is a powerful and useful equation because it relates pressure changes to velocity and elevation changes along a streamline. However, it gives correct results only when applied to a flow situation where all four of the restrictions are reasonable. Keep the restrictions firmly in mind whenever you consider using the Bernoulli equation. (In general, the Bernoulli constant in Eq. 6.8 has different values along different streamlines.[3])

*Derivation Using Rectangular Coordinates

The vector form of Euler's equation, Eq. 6.1, also can be integrated along a streamline. We shall restrict the derivation to steady flow; thus, the end result of our effort should be Eq. 6.7.

For steady flow, Euler's equation in rectangular coordinates can be expressed as

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = (\vec{V}\cdot\nabla)\vec{V} = -\frac{1}{\rho}\nabla p - g\hat{k} \tag{6.9}$$

For steady flow the velocity field is given by $\vec{V} = \vec{V}(x, y, z)$. The streamlines are lines drawn in the flow field tangent to the velocity vector at every point. Recall again that for steady flow, streamlines, pathlines, and streaklines coincide. The motion of a particle along a streamline is governed by Eq. 6.9. During time interval dt the particle has vector displacement $d\vec{s}$ along the streamline.

If we take the dot product of the terms in Eq. 6.9 with displacement $d\vec{s}$ along the streamline, we obtain a scalar equation relating pressure, speed, and elevation along the streamline. Taking the dot product of $d\vec{s}$ with Eq. 6.9 gives

$$(\vec{V}\cdot\nabla)\vec{V}\cdot d\vec{s} = -\frac{1}{\rho}\nabla p\cdot d\vec{s} - g\hat{k}\cdot d\vec{s} \tag{6.10}$$

where

$$d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k} \qquad (\text{along } s)$$

Now we evaluate each of the three terms in Eq. 6.10, starting on the right,

$$-\frac{1}{\rho}\nabla p\cdot d\vec{s} = -\frac{1}{\rho}\left[\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right]\cdot[dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= -\frac{1}{\rho}\left[\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right] \qquad (\text{along } s)$$

$$-\frac{1}{\rho}\nabla p\cdot d\vec{s} = -\frac{1}{\rho}dp \qquad (\text{along } s)$$

[3] For the case of irrotational flow, the constant has a single value throughout the entire flow field (Section 6-7).
*This section may be omitted without loss of continuity in the text material.

and

$$-g\hat{k}\cdot d\vec{s} = -g\hat{k}\cdot[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$$
$$= -g\,dz \qquad (\text{along } s)$$

Using a vector identity,[4] we can write the third term as

$$(\vec{V}\cdot\nabla)\vec{V}\cdot d\vec{s} = \left[\tfrac{1}{2}\nabla(\vec{V}\cdot\vec{V}) - \vec{V}\times(\nabla\times\vec{V})\right]\cdot d\vec{s}$$
$$= \left\{\tfrac{1}{2}\nabla(\vec{V}\cdot\vec{V})\right\}\cdot d\vec{s} - \left\{\vec{V}\times(\nabla\times\vec{V})\right\}\cdot d\vec{s}$$

The last term on the right side of this equation is zero, since $\vec{V}$ is parallel to $d\vec{s}$. Consequently,

$$(\vec{V}\cdot\nabla)\vec{V}\cdot d\vec{s} = \tfrac{1}{2}\nabla(\vec{V}\cdot\vec{V})\cdot d\vec{s} = \tfrac{1}{2}\nabla(V^2)\cdot d\vec{s} \quad (\text{along } s)$$
$$= \frac{1}{2}\left[\hat{i}\,\frac{\partial V^2}{\partial x} + \hat{j}\,\frac{\partial V^2}{\partial y} + \hat{k}\,\frac{\partial V^2}{\partial z}\right]\cdot[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$$
$$= \frac{1}{2}\left[\frac{\partial V^2}{\partial x}dx + \frac{\partial V^2}{\partial y}dy + \frac{\partial V^2}{\partial z}dz\right]$$
$$(\vec{V}\cdot\nabla)\vec{V}\cdot d\vec{s} = \frac{1}{2}d(V^2) \quad (\text{along } s)$$

Substituting these three terms into Eq. 6.10 yields

$$\frac{dp}{\rho} + \frac{1}{2}d(V^2) + g\,dz = 0 \quad (\text{along } s)$$

Integrating this equation, we obtain

$$\int\frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \qquad (\text{along } s)$$

If the density is constant, we obtain the Bernoulli equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

As expected, we see that the last two equations are identical to Eqs. 6.7 and 6.8 derived previously using streamline coordinates. The Bernoulli equation, derived using rectangular coordinates, is still subject to the restrictions: (1) steady flow, (2) incompressible flow, (3) frictionless flow, and (4) flow along a streamline.

Static, Stagnation, and Dynamic Pressures

The pressure, p, which we have used in deriving the Bernoulli equation, Eq. 6.8, is the thermodynamic pressure; it is commonly called the *static pressure.* The static pressure is the pressure seen by the fluid particle as it moves (so it is something of a misnomer!)—

[4] The vector identity

$$(\vec{V}\cdot\nabla)\vec{V} = \tfrac{1}{2}\nabla(\vec{V}\cdot\vec{V}) - \vec{V}\times(\nabla\times\vec{V})$$

may be verified by expanding each side into components.

we also have the stagnation and dynamic pressures, which we will define shortly. How do we measure the pressure in a fluid in motion?

In Section 6-2 we showed that there is no pressure variation normal to straight streamlines. This fact makes it possible to measure the static pressure in a flowing fluid using a wall pressure "tap," placed in a region where the flow streamlines are straight, as shown in Fig. 6.2*a*. The pressure tap is a small hole, drilled carefully in the wall, with its axis perpendicular to the surface. If the hole is perpendicular to the duct wall and free from burrs, accurate measurements of static pressure can be made by connecting the tap to a suitable pressure-measuring instrument [1].

In a fluid stream far from a wall, or where streamlines are curved, accurate static pressure measurements can be made by careful use of a static pressure probe, shown in Fig. 6.2*b*. Such probes must be designed so that the measuring holes are placed correctly with respect to the probe tip and stem to avoid erroneous results [2]. In use, the measuring section must be aligned with the local flow direction.

Static pressure probes, such as that shown in Fig 6.2*b*, and in a variety of other forms, are available commercially in sizes as small as 1.5 mm ($\frac{1}{16}$ in.) in diameter [3].

The *stagnation pressure* is obtained when a flowing fluid is decelerated to zero speed by a frictionless process. For incompressible flow, the Bernoulli equation can be used to relate changes in speed and pressure along a streamline for such a process. Neglecting elevation differences, Eq. 6.8 becomes

$$\frac{p}{\rho} + \frac{V^2}{2} = \text{constant}$$

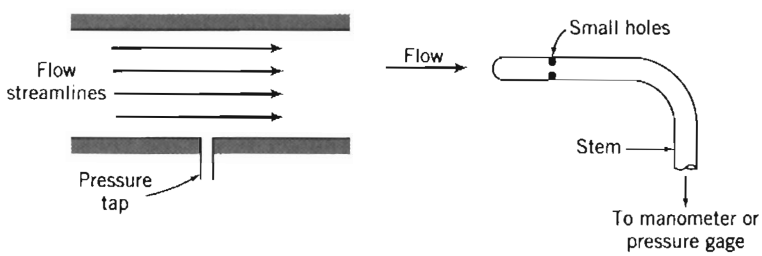
If the static pressure is p at a point in the flow where the speed is V, then the stagnation pressure, p_0, where the stagnation speed, V_0, is zero, may be computed from

$$\frac{p_0}{\rho} + \overbrace{\cancel{\frac{V_0^2}{2}}}^{=0} = \frac{p}{\rho} + \frac{V^2}{2}$$

or

$$p_0 = p + \frac{1}{2}\rho V^2 \tag{6.11}$$

Equation 6.11 is a mathematical statement of the definition of stagnation pressure, valid for incompressible flow. The term $\frac{1}{2}\rho V^2$ generally is called the *dynamic pressure*. Equation 6.11 states that the stagnation (or *total*) pressure equals the static pressure plus the dynamic pressure. One way to picture the three pressures is to imagine you are standing in a steady wind holding up your hand: The static pressure will be atmospheric pressure; the larger pressure you feel at the center of your hand will be the stagnation pressure; and the buildup of pressure will be the dynamic pressure.



(*a*) Wall pressure tap (*b*) Static pressure probe

Fig. 6.2 Measurement of static pressure.

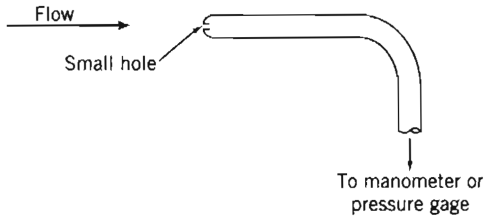


Fig. 6.3 Measurement of stagnation pressure.

Solving Eq. 6.11 for the speed,

$$V = \sqrt{\frac{2(p_0 - p)}{\rho}} \tag{6.12}$$

Thus, if the stagnation pressure and the static pressure could be measured at a point, Eq. 6.12 would give the local flow speed.

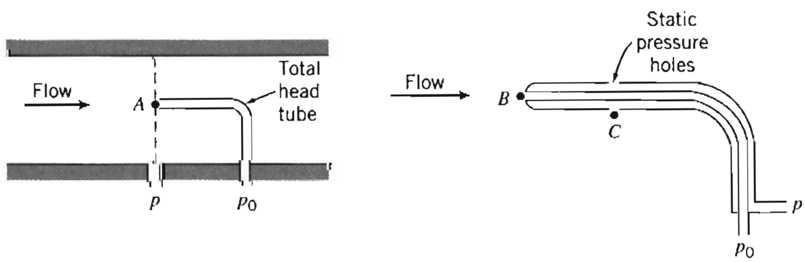
Stagnation pressure is measured in the laboratory using a probe with a hole that faces directly upstream as shown in Fig. 6.3. Such a probe is called a stagnation pressure probe, or pitot (pronounced *pea-toe*) tube. Again, the measuring section must be aligned with the local flow direction.

We have seen that static pressure at a point can be measured with a static pressure tap or probe (Fig. 6.2). If we knew the stagnation pressure at the same point, then the flow speed could be computed from Eq. 6.12. Two possible experimental setups are shown in Fig. 6.4.

In Fig. 6.4*a*, the static pressure corresponding to point A is read from the wall static pressure tap. The stagnation pressure is measured directly at A by the total head tube, as shown. (The stem of the total head tube is placed downstream from the measurement location to minimize disturbance of the local flow.)

Two probes often are combined, as in the pitot-static tube shown in Fig. 6.4*b*. The inner tube is used to measure the stagnation pressure at point B, while the static pressure at C is sensed using the small holes in the outer tube. In flow fields where the static pressure variation in the streamwise direction is small, the pitot-static tube may be used to infer the speed at point B in the flow by assuming $p_B = p_C$ and using Eq. 6.12. (Note that when $p_B \neq p_C$, this procedure will give erroneous results.)

Remember that the Bernoulli equation applies only for incompressible flow (Mach number $M \leq 0.3$). The definition and calculation of the stagnation pressure for compressible flow will be discussed in Section 11-3.



(*a*) Total head tube used with wall static tap

(*b*) Pitot-static tube

Fig. 6.4 Simultaneous measurement of stagnation and static pressures.

EXAMPLE 6.2 Pitot Tube

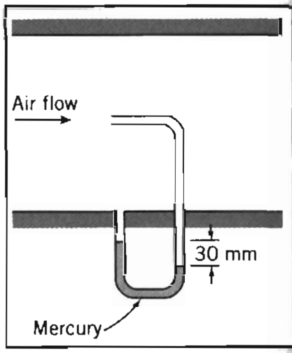
A pitot tube is inserted in an air flow (at STP) to measure the flow speed. The tube is inserted so that it points upstream into the flow and the pressure sensed by the tube is the stagnation pressure. The static pressure is measured at the same location in the flow, using a wall pressure tap. If the pressure difference is 30 mm of mercury, determine the flow speed.

EXAMPLE PROBLEM 6.2

GIVEN: A pitot tube inserted in a flow as shown. The flowing fluid is air and the manometer liquid is mercury.

FIND: The flow speed.

SOLUTION:



Governing equation: $\dfrac{p}{\rho} + \dfrac{V^2}{2} + gz = \text{constant}$

Assumptions:
(1) Steady flow.
(2) Incompressible flow.
(3) Flow along a streamline.
(4) Frictionless deceleration along stagnation streamline.

Writing Bernoulli's equation along the stagnation streamline (with $\Delta z = 0$) yields

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$$

p_0 is the stagnation pressure at the tube opening where the speed has been reduced, without friction, to zero. Solving for V gives

$$V = \sqrt{\frac{2(p_0 - p)}{\rho_{\text{air}}}}$$

From the diagram,

$$p_0 - p = \rho_{\text{Hg}}\, gh = \rho_{\text{H}_2\text{O}}\, g\, h\, (\text{SG}_{\text{Hg}})$$

and

$$V = \sqrt{\frac{2\rho_{\text{H}_2\text{O}}\, gh(\text{SG}_{\text{Hg}})}{\rho_{\text{air}}}}$$

$$= \sqrt{2 \times \frac{1000\ \text{kg}}{\text{m}^3} \times \frac{9.81\ \text{m}}{\text{s}^2} \times 30\ \text{mm} \times 13.6 \times \frac{\text{m}^3}{1.23\ \text{kg}} \times \frac{\text{m}}{1000\ \text{mm}}}$$

$V = 80.8$ m/s $\longleftarrow$ V

At $T = 20°\text{C}$, the speed of sound in air is 343 m/s. Hence, $M = 0.236$ and the assumption of incompressible flow is valid.

> This problem illustrates use of a pitot tube to determine flow speed. Pitot (or pitot-static) tubes are often placed on the exterior of aircraft to indicate air speed relative to the aircraft, and hence aircraft speed relative to the air.

Applications

The Bernoulli equation can be applied between any two points on a streamline provided that the other three restrictions are satisfied. The result is

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 \tag{6.13}$$

where subscripts 1 and 2 represent any two points on a streamline. Applications of Eqs. 6.8 and 6.13 to typical flow problems are illustrated in Example Problems 6.3 through 6.5.

In some situations, the flow appears unsteady from one reference frame, but steady from another, which translates with the flow. Since the Bernoulli equation was derived by integrating Newton's second law for a fluid particle, it can be applied in any inertial reference frame (see the discussion of translating frames in Section 4-4). The procedure is illustrated in Example Problem 6.6.

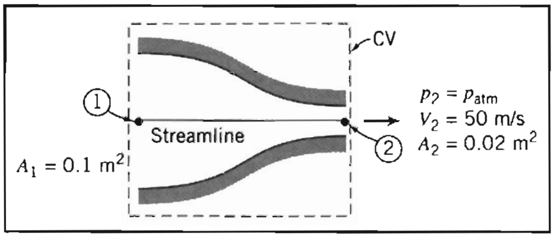
EXAMPLE 6.3 Nozzle Flow

Air flows steadily at low speed through a horizontal *nozzle* (by definition a device for accelerating a flow), discharging to atmosphere. The area at the nozzle inlet is 0.1 m^2. At the nozzle exit, the area is 0.02 m^2. Determine the gage pressure required at the nozzle inlet to produce an outlet speed of 50 m/s.

EXAMPLE PROBLEM 6.3

GIVEN: Flow through a nozzle, as shown.

FIND: $p_1 - p_{atm}$.



SOLUTION:

Governing equations:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$= 0(1)$$

$$\cancel{\frac{\partial}{\partial t}}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assumptions:
(1) Steady flow.
(2) Incompressible flow.
(3) Frictionless flow.
(4) Flow along a streamline.
(5) $z_1 = z_2$.
(6) Uniform flow at sections 1 and 2.

The maximum speed of 50 m/s is well below 100 m/s, which corresponds to Mach number $M \approx 0.3$ in standard air. Hence, the flow may be treated as incompressible.

Apply the Bernoulli equation along a streamline between points 1 and 2 to evaluate p_1. Then

$$p_1 - p_{atm} = p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Apply the continuity equation to determine V_1,

$$(-\rho V_1 A_1)+(\rho V_2 A_2) = 0 \qquad \text{or} \qquad V_1 A_1 = V_2 A_2$$

so that

$$V_1 = V_2 \frac{A_2}{A_1} = \frac{50\ \text{m}}{\text{s}} \times \frac{0.02\ \text{m}^2}{0.1\ \text{m}^2} = 10\ \text{m/s}$$

For air at standard conditions, $\rho = 1.23\ \text{kg/m}^3$. Then

$$p_1 - p_{\text{atm}} = \frac{\rho}{2}(V_2^2 - V_1^2)$$

$$= \frac{1}{2} \times 1.23 \frac{\text{kg}}{\text{m}^3}\left[(50)^2 \frac{\text{m}^2}{\text{s}^2} - (10)^2 \frac{\text{m}^2}{\text{s}^2}\right]\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$p_1 - p_{\text{atm}} = 1.48\ \text{kPa}$$

Notes:

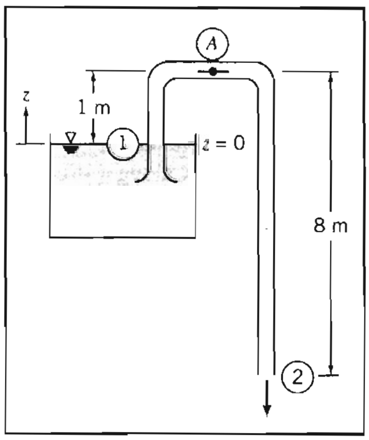
- ✓ This problem illustrates a typical application of the Bernoulli equation.
- ✓ The streamlines must be straight at the inlet and exit in order to have uniform pressures at those locations.

EXAMPLE 6.4 Flow through a Siphon

A U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface; the tube outlet is 7 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine (after listing the necessary assumptions) the speed of the free jet and the minimum absolute pressure of the water in the bend.

EXAMPLE PROBLEM 6.4

GIVEN: Water flowing through a siphon as shown.



FIND: (a) Speed of water leaving as a free jet.
(b) Pressure at point Ⓐ in the flow.

SOLUTION:

Governing equation: $\dfrac{p}{\rho} + \dfrac{V^2}{2} + gz = \text{constant}$

Assumptions:
(1) Neglect friction.
(2) Steady flow.
(3) Incompressible flow.
(4) Flow along a streamline.
(5) Reservoir is large compared with pipe.

Apply the Bernoulli equation between points ① and ②.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Since $\text{area}_{\text{reservoir}} \gg \text{area}_{\text{pipe}}$, then $V_1 \approx 0$. Also $p_1 = p_2 = p_{\text{atm}}$, so

$$gz_1 = \frac{V_2^2}{2} + gz_2 \quad \text{and} \quad V_2^2 = 2g(z_1 - z_2)$$

$$V_2 = \sqrt{2g(z_1 - z_2)} = \sqrt{2 \times 9.81\,\frac{\text{m}}{\text{s}^2} \times 7\,\text{m}} = 11.7\,\text{m/s} \longleftarrow \qquad V_2$$

To determine the pressure at location Ⓐ, we write the Bernoulli equation between ① and Ⓐ.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A$$

Again $V_1 \approx 0$ and from conservation of mass $V_A = V_2$. Hence

$$\frac{p_A}{\rho} = \frac{p_1}{\rho} + gz_1 - \frac{V_2^2}{2} - gz_A = \frac{p_1}{\rho} + g(z_1 - z_A) - \frac{V_2^2}{2}$$

$$p_A = p_1 + \rho g(z_1 - z_A) - \rho\frac{V_2^2}{2}$$

$$= 1.01 \times 10^5\,\frac{\text{N}}{\text{m}^3} + 999\,\frac{\text{kg}}{\text{m}^3} \times 9.81\,\frac{\text{m}}{\text{s}^2} \times (-1\,\text{m})\,\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$-\frac{1}{2} \times 999\,\frac{\text{kg}}{\text{m}^3} \times (11.7)^2\,\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$p_A = 22.8\,\text{kPa (abs) or } -78.5\,\text{kPa (gage)} \longleftarrow \qquad p_A$$

Notes:
- ✓ This problem illustrates an application of the Bernoulli equation that includes elevation changes.
- ✓ Always take care when neglecting friction in any internal flow. In this problem, neglecting friction is reasonable if the pipe is smooth-surfaced and is relatively short. In Chapter 8 we will study frictional effects in internal flows.

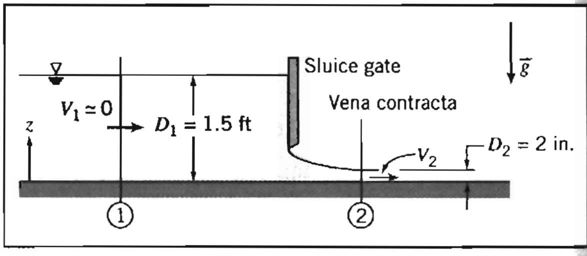
EXAMPLE 6.5 Flow under a Sluice Gate

Water flows under a sluice gate on a horizontal bed at the inlet to a flume. Upstream from the gate, the water depth is 1.5 ft and the speed is negligible. At the vena contracta downstream from the gate, the flow streamlines are straight and the depth is 2 in. Determine the flow speed downstream from the gate and the discharge in cubic feet per second per foot of width.

EXAMPLE PROBLEM 6.5

GIVEN: Flow of water under a sluice gate.

FIND: (a) V_2.
(b) Q in ft^3/s/ft of width.



SOLUTION:

Under the assumptions listed below, the flow satisfies all conditions necessary to apply the Bernoulli equation. The question is, what streamline do we use?

Governing equation: $$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assumptions: (1) Steady flow.
(2) Incompressible flow.
(3) Frictionless flow.
(4) Flow along a streamline.
(5) Uniform flow at each section.
(6) Hydrostatic pressure distribution.

If we consider the streamline that runs along the bottom of the channel ($z = 0$), because of assumption 6 the pressures at ① and ② are

$$p_1 = p_{atm} + \rho g D_1 \quad \text{and} \quad p_2 = p_{atm} + \rho g D_2$$

so that the Bernoulli equation for this streamline is

$$\frac{(p_{atm} + \rho g D_1)}{\rho} + \frac{V_1^2}{2} = \frac{(p_{atm} + \rho g D_2)}{\rho} + \frac{V_2^2}{2}$$

or

$$\frac{V_1^2}{2} + gD_1 = \frac{V_2^2}{2} + gD_2 \qquad (1)$$

On the other hand, consider the streamline that runs along the free surface on both sides of the gate. For this streamline

$$\frac{p_{atm}}{\rho} + \frac{V_1^2}{2} + gD_1 = \frac{p_{atm}}{\rho} + \frac{V_2^2}{2} + gD_2$$

or

$$\frac{V_1^2}{2} + gD_1 = \frac{V_2^2}{2} + gD_2 \qquad (1)$$

We have arrived at the same equation (Eq. 1) for the streamline at the bottom and the streamline at the free surface, implying the Bernoulli constant is the same for both streamlines. We will see in Section 6-6 that this flow is one of a family of flows for which this is the case. Solving for V_2 yields

$$V_2 = \sqrt{2g(D_1 - D_2) + V_1^2}$$

But $V_1^2 \approx 0$, so

$$V_2 = \sqrt{2g(D_1 - D_2)} = \sqrt{2 \times 32.2\,\frac{\text{ft}}{\text{s}^2}\left(1.5\,\text{ft} - 2\,\text{in.} \times \frac{\text{ft}}{12\,\text{in.}}\right)}$$

$$V_2 = 9.27\,\text{ft/s} \qquad V_2$$

For uniform flow, $Q = VA = VDw$, or

$$\frac{Q}{w} = VD = V_2 D_2 = 9.27\,\frac{\text{ft}}{\text{s}} \times 2\,\text{in.} \times \frac{\text{ft}}{12\,\text{in.}} = 1.55\,\text{ft}^2/\text{s}$$

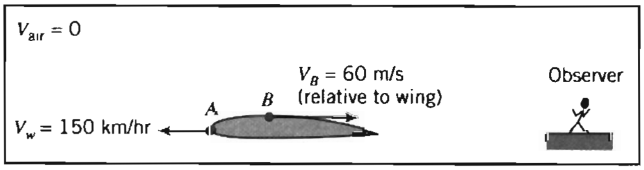
$$\frac{Q}{w} = 1.55\,\text{ft}^3/\text{s/foot of width} \qquad \frac{Q}{w}$$

EXAMPLE 6.6 Bernoulli Equation in Translating Reference Frame

A light plane flies at 150 km/hr in standard air at an altitude of 1000 m. Determine the stagnation pressure at the leading edge of the wing. At a certain point close to the wing, the air speed *relative* to the wing is 60 m/s. Compute the pressure at this point.

EXAMPLE PROBLEM 6.6

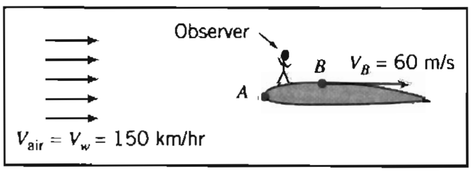
GIVEN: Aircraft in flight at 150 km/hr at 1000 m altitude in standard air.



FIND: Stagnation pressure, p_{0_A}, at point A and static pressure, p_B, at point B.

SOLUTION:

Flow is unsteady when observed from a fixed frame, that is, by an observer on the ground. However, an observer *on* the wing sees the following steady flow:



At $z = 1000$ m in standard air, the temperature is 281 K and the speed of sound is 336 m/s. Hence at point B, $M_B = V_B/c = 0.178$. This is less than 0.3, so the flow may be treated as incompressible. Thus the Bernoulli equation can be applied along a streamline in the moving observer's inertial reference frame.

Governing equation: $$\frac{p_{air}}{\rho} + \frac{V_{air}^2}{2} + gz_{air} = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A = \frac{p_B}{\rho} + \frac{V_B^2}{2} + gz_B$$

Assumptions: (1) Steady flow.
(2) Incompressible flow ($V < 100$ m/s).
(3) Frictionless flow.
(4) Flow along a streamline.
(5) Neglect Δz.

Values for pressure and density may be found from Table A.3. Thus, at 1000 m, $p/p_{SL} = 0.8870$ and $\rho/\rho_{SL} = 0.9075$. Consequently,

$$p = 0.8870 p_{SL} = 0.8870 \times 1.01 \times 10^5 \frac{\text{N}}{\text{m}^2} = 8.96 \times 10^4 \text{ N/m}^2$$

and

$$\rho = 0.9075 \rho_{SL} = 0.9075 \times 1.23 \frac{\text{kg}}{\text{m}^3} = 1.12 \text{ kg/m}^3$$

Since the speed is $V_A = 0$ at the stagnation point,

$$p_{0_A} = p_{\text{air}} + \frac{1}{2}\rho V_{\text{air}}^2$$

$$= 8.96 \times 10^4 \frac{\text{N}}{\text{m}^2} + \frac{1}{2} \times 1.12 \frac{\text{kg}}{\text{m}^3}\left(150 \frac{\text{km}}{\text{hr}} \times 1000 \frac{\text{m}}{\text{km}} \times \frac{\text{hr}}{3600 \text{ s}}\right)^2 \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

$$p_{0_A} = 90.6 \text{ kPa (abs)} \longleftarrow p_{0_A}$$

Solving for the static pressure at B, we obtain

$$p_B = p_{\text{air}} + \frac{1}{2}\rho (V_{\text{air}}^2 - V_B^2)$$

$$p_B = 8.96 \times 10^4 \frac{\text{N}}{\text{m}^2} + \frac{1}{2} \times 1.12 \frac{\text{kg}}{\text{m}^3}\left[\left(150 \frac{\text{km}}{\text{hr}} \times 1000 \frac{\text{m}}{\text{km}} \times \frac{\text{hr}}{3600 \text{ s}}\right)^2 - (60)^2 \frac{\text{m}^2}{\text{s}^2}\right] \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}}$$

$$p_B = 88.6 \text{ kPa (abs)} \longleftarrow p_B$$

Cautions on Use of the Bernoulli Equation

In Example Problems 6.3 through 6.6, we have seen several situations where the Bernoulli equation may be applied because the restrictions on its use led to a reasonable flow model. However, in some situations you might be tempted to apply the Bernoulli equation where the restrictions are not satisfied. Some subtle cases that violate the restrictions are discussed briefly in this section.

Example Problem 6.3 examined flow in a nozzle. In a *subsonic nozzle* (a converging section) the pressure drops, accelerating a flow. Because the pressure drops and the walls of the nozzle converge, there is no flow separation from the walls and the boundary layer remains thin. In addition, a nozzle is usually relatively short so frictional effects are not significant. All of this leads to the conclusion that the Bernoulli equation is suitable for use for subsonic nozzles.

Sometimes we need to decelerate a flow. This can be accomplished using a *subsonic diffuser* (a diverging section), or by using a sudden expansion (e.g., from a pipe into a reservoir). In these devices the flow decelerates because of an adverse pressure gradient. As we discussed in Section 2-6, an adverse pressure gradient tends to lead to rapid growth of the boundary layer and its separation.[5] Hence, we should be careful in applying the Bernoulli equation in such devices—at best, it will be an approximation. Because of area blockage caused by boundary-layer growth, pressure rise in actual diffusers always is less than that predicted for inviscid one-dimensional flow.

The Bernoulli equation was a reasonable model for the siphon of Example Problem 6.4 because the entrance was well rounded, the bends were gentle, and the overall length was short. Flow separation, which can occur at inlets with sharp corners and in abrupt bends, causes the flow to depart from that predicted by a one-dimensional model and the Bernoulli equation. Frictional effects would not be negligible if the tube were long.

Example Problem 6.5 presented an open-channel flow analogous to that in a nozzle, for which the Bernoulli equation is a good flow model. The hydraulic jump[6] is an example of an open-channel flow with adverse pressure gradient. Flow through a hydraulic jump is mixed violently, making it impossible to identify streamlines. Thus the Bernoulli equation cannot be used to model flow through a hydraulic jump.

The Bernoulli equation cannot be applied *through* a machine such as a propeller, pump, turbine, or windmill. The equation was derived by integrating along a stream tube (Section 4-4) or a streamline (Section 6-3) in the absence of moving surfaces such as blades or vanes. It is impossible to have locally steady flow or to identify streamlines during flow through a machine. Hence, while the Bernoulli equation may be applied between points *before* a machine, or between points *after* a machine (assuming its restrictions are satisfied), it cannot be applied *through* the machine. (In effect, a machine will change the value of the Bernoulli constant.)

Finally, compressibility must be considered for flow of gases. Density changes caused by dynamic compression due to motion may be neglected for engineering purposes if the local Mach number remains below about $M \approx 0.3$, as noted in Example Problems 6.3 and 6.6. Temperature changes can cause significant changes in density of a gas, even for low-speed flow. Thus the Bernoulli equation could not be applied to air flow through a heating element (e.g., of a hand-held hair dryer) where temperature changes are significant.

6-4 THE BERNOULLI EQUATION INTERPRETED AS AN ENERGY EQUATION

The Bernoulli equation, Eq. 6.8, was obtained by integrating Euler's equation along a streamline for steady, incompressible, frictionless flow. Thus Eq. 6.8 was derived from the momentum equation for a fluid particle.

An equation identical in form to Eq. 6.8 (although requiring very different restrictions) may be obtained from the first law of thermodynamics. Our objective in this section is to reduce the energy equation to the form of the Bernoulli equation given by Eq. 6.8. Having arrived at this form, we then compare the restrictions on the two equations to help us understand more clearly the restrictions on the use of Eq. 6.8.

Consider steady flow in the absence of shear forces. We choose a control volume bounded by streamlines along its periphery. Such a boundary, shown in Fig. 6.5, often is called a *stream tube*.

[5] See the NCFMF video *Flow Visualization*.

[6] See the NCFMF videos *Waves in Fluids* and *Stratified Flow* for examples of this behavior.

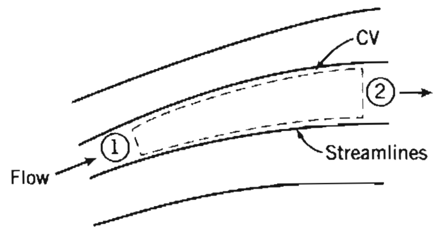


Fig. 6.5 Flow through a stream tube.

Basic equation:

$$\dot{Q} - \underbrace{\cancel{\dot{W}_s}}_{=0(1)} - \underbrace{\cancel{\dot{W}_{\text{shear}}}}_{=0(2)} - \underbrace{\cancel{\dot{W}_{\text{other}}}}_{=0(3)} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(4)}\int_{\text{CV}} e\,\rho\,dV + \int_{\text{CS}} (e + pv)\,\rho\vec{V}\cdot d\vec{A} \tag{4.56}$$

$$e = u + \frac{V^2}{2} + gz$$

Restrictions: (1) $\dot{W}_s = 0$.

(2) $\dot{W}_{\text{shear}} = 0$.

(3) $\dot{W}_{\text{other}} = 0$.

(4) Steady flow.

(5) Uniform flow and properties at each section.

(Remember that here v represents the specific volume, and u represents the specific internal energy, not velocity!) Under these restrictions, Eq. 4.56 becomes

$$\left(u_1 + p_1v_1 + \frac{V_1^2}{2} + gz_1\right)(-\rho_1V_1A_1) + \left(u_2 + p_2v_2 + \frac{V_2^2}{2} + gz_2\right)(\rho_2V_2A_2) - \dot{Q} = 0$$

But from continuity under these restrictions,

$$\underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(4)}\int_{\text{CV}} \rho\,dV + \int_{\text{CS}} \rho\,\vec{V}\cdot d\vec{A} = 0$$

or

$$(-\rho_1V_1A_1) + (\rho_2V_2A_2) = 0$$

That is,

$$\dot{m} = \rho_1V_1A_1 = \rho_2V_2A_2$$

Also

$$\dot{Q} = \frac{\delta Q}{dt} = \frac{\delta Q}{dm}\frac{dm}{dt} = \frac{\delta Q}{dm}\dot{m}$$

Thus, from the energy equation,

$$\left[\left(p_2v_2 + \frac{V_2^2}{2} + gz_2\right) - \left(p_1v_1 + \frac{V_1^2}{2} + gz_1\right)\right]\dot{m} + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)\dot{m} = 0$$

or

$$p_1 v_1 + \frac{V_1^2}{2} + gz_1 = p_2 v_2 + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)$$

Under the additional assumption (6) of incompressible flow, $v_1 = v_2 = 1/\rho$ and hence

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right) \tag{6.14}$$

Equation 6.14 would reduce to the Bernoulli equation if the term in parentheses were zero. Thus, under the further restriction,

$$(7) \quad (u_2 - u_1 - \delta Q/dm) = 0$$

the energy equation reduces to

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

or

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{6.15}$$

Equation 6.15 is identical in form to the Bernoulli equation, Eq. 6.8. The Bernoulli equation was derived from momentum considerations (Newton's second law), and is valid for steady, incompressible, frictionless flow along a streamline. Equation 6.15 was obtained by applying the first law of thermodynamics to a stream tube control volume, subject to restrictions 1 through 7 above. Thus the Bernoulli equation (Eq. 6.8) and the identical form of the energy equation (Eq. 6.15) were developed from entirely different models, coming from entirely different basic concepts, and involving different restrictions.

Note that restriction 7 was necessary to obtain the Bernoulli equation from the first law of thermodynamics. This restriction can be satisfied if $\delta Q/dm$ is zero (there is no heat transfer to the fluid) and $u_2 = u_1$ (there is no change in the internal thermal energy of the fluid). The restriction also is satisfied if $(u_2 - u_1)$ and $\delta Q/dm$ are nonzero provided that the two terms are equal. That this is true for incompressible frictionless flow is shown in Example Problem 6.7.

EXAMPLE 6.7 Internal Energy and Heat Transfer in Frictionless Incompressible Flow

Consider frictionless, incompressible flow with heat transfer. Show that

$$u_2 - u_1 = \frac{\delta Q}{dm}$$

EXAMPLE PROBLEM 6.7

GIVEN: Frictionless, incompressible flow with heat transfer.

SHOW: $u_2 - u_1 = \dfrac{\delta Q}{dm}$.

SOLUTION:

In general, internal energy can be expressed as $u = u(T, v)$. For incompressible flow, v = constant, and $u = u(T)$. Thus the thermodynamic state of the fluid is determined by the single thermodynamic property, T. For any process, the internal energy change, $u_2 - u_1$, depends only on the temperatures at the end states.

From the Gibbs equation, $Tds = du + p\,dv$, valid for a pure substance undergoing any process, we obtain

$$Tds = du$$

for incompressible flow, since $dv = 0$. Since the internal energy change, du, between specified end states, is independent of the process, we take a reversible process, for which $Tds = d(\delta Q/dm) = du$. Therefore,

$$u_2 - u_1 = \frac{\delta Q}{dm}$$

For the special case considered in this section, it is true that the first law of thermodynamics reduces to the Bernoulli equation. Each term in Eq. 6.15 has dimensions of energy per unit mass (we sometimes refer to the three terms in the equation as the "pressure" energy, kinetic energy, and potential energy per unit mass of the fluid). It is not surprising that Eq. 6.15 contains energy terms—after all, we used the first law of thermodynamics in deriving it. How did we end up with the same energy-like terms in the Bernoulli equation, which we derived from the momentum equation? The answer is because we integrated the momentum equation (which involves force terms) along a streamline (which involves distance), and by doing so ended up with work or energy terms (work being defined as force times distance): The work of gravity and pressure forces leads to a kinetic energy change (which came from integrating momentum over distance). In this context, we can think of the Bernoulli equation as a *mechanical energy balance*—the mechanical energy ("pressure" plus potential plus kinetic) will be constant. We must always bear in mind that for the Bernoulli equation to be valid along a streamline requires an incompressible inviscid flow, in addition to steady flow. If we had density changes they would continuously allow conversion of any or all of the mechanical energy forms to internal thermal energy, and vice versa. Friction always converts mechanical energy to thermal energy (appearing either as a gain of internal thermal energy or as heat generation, or both). In the absence of density changes and friction, the mechanisms linking the mechanical and internal thermal energy do not exist, and restriction 7 holds—any internal thermal energy changes will result only from a heat transfer process and be independent of the fluid mechanics, and the thermodynamic and mechanical energies will be uncoupled.

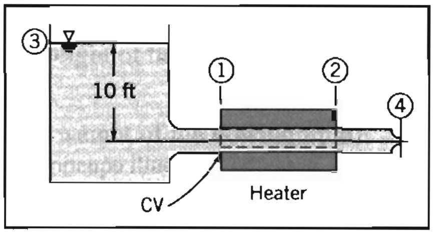
In summary, when the conditions are satisfied for the Bernoulli equation to be valid, we can consider separately the mechanical energy and the internal thermal energy of a fluid particle (this is illustrated in Example Problem 6.8); when they are not satisfied, there will be an interaction between these energies, the Bernoulli equation becomes invalid, and we must use the full first law of thermodynamics.

EXAMPLE 6.8 Frictionless Flow with Heat Transfer

Water flows steadily from a large open reservoir through a short length of pipe and a nozzle with cross-sectional area $A = 0.864$ in.^{2} A well-insulated 10 kW heater surrounds the pipe. Find the temperature rise of the water.

EXAMPLE PROBLEM 6.8

GIVEN: Water flows from a large reservoir through the system shown and discharges to atmospheric pressure. The heater is 10 kW; $A_4 = 0.864\ \text{in.}^2$



FIND: The temperature rise of the water between points ① and ②.

SOLUTION:

Governing equations: $\dfrac{p}{\rho} + \dfrac{V^2}{2} + gz = \text{constant}$

$$\overset{=0(1)}{\cancel{\frac{\partial}{\partial t}\int_{\text{CV}}}} \rho\, d\forall + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\dot{Q} - \overset{=0(4)}{\cancel{\dot{W}_s}} - \overset{=0(4)}{\cancel{\dot{W}_{\text{shear}}}} = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}\int_{\text{CV}}}} e\, \rho\, d\forall + \int_{\text{CS}} \left(u + pv + \frac{V^2}{2} + gz\right) \rho \vec{V} \cdot d\vec{A}$$

Assumptions:
(1) Steady flow.
(2) Frictionless flow.
(3) Incompressible flow.
(4) No shaft work, no shear work.
(5) Flow along a streamline.

Under the assumptions listed, the first law of thermodynamics for the CV shown becomes

$$\dot{Q} = \int_{\text{CS}} \left(u + pv + \frac{V^2}{2} + gz\right) \rho \vec{V} \cdot d\vec{A}$$

$$= \int_{A_1} \left(u + pv + \frac{V^2}{2} + gz\right) \rho \vec{V} \cdot d\vec{A} + \int_{A_2} \left(u + pv + \frac{V^2}{2} + gz\right) \rho \vec{V} \cdot d\vec{A}$$

For uniform properties at ① and ②

$$\dot{Q} = -(\rho V_1 A_1)\left(u_1 + p_1 v + \frac{V_1^2}{2} + gz_1\right) + (\rho V_2 A_2)\left(u_2 + p_2 v + \frac{V_2^2}{2} + gz_2\right)$$

From conservation of mass, $\rho V_1 A_1 = \rho V_2 A_2 = \dot{m}$, so

$$\dot{Q} = \dot{m}\left[u_2 - u_1 + \left(\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2\right) - \left(\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1\right)\right]$$

For frictionless, incompressible, steady flow, along a streamline,

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Therefore,

$$\dot{Q} = \dot{m}(u_2 - u_1)$$

Since, for an incompressible fluid, $u_2 - u_1 = c(T_2 - T_1)$, then

$$T_2 - T_1 = \frac{\dot{Q}}{\dot{m}c}$$

From continuity,

$$\dot{m} = \rho V_4 A_4$$

To find V_4, write the Bernoulli equation between the free surface at ③ and point ④.

$$\frac{p_3}{\rho} + \frac{V_3^2}{2} + gz_3 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + gz_4$$

Since $p_3 = p_4$ and $V_3 \approx 0$, then

$$V_4 = \sqrt{2g(z_3 - z_4)} = \sqrt{2 \times \frac{32.2\ \text{ft}}{\text{s}^2} \times 10\ \text{ft}} = 25.4\ \text{ft/s}$$

and

$$\dot{m} = \rho V_4 A_4 = 1.94\ \frac{\text{slug}}{\text{ft}^3} \times 25.4\ \frac{\text{ft}}{\text{s}} \times 0.864\ \text{in.}^2 \times \frac{\text{ft}^2}{144\ \text{in.}^2} = 0.296\ \text{slug/s}$$

Assuming no heat loss to the surroundings, we obtain

$$T_2 - T_1 = \frac{\dot{Q}}{\dot{m}c} = 10\ \text{kW} \times 3413\ \frac{\text{Btu}}{\text{kW}\cdot\text{hr}} \times \frac{\text{hr}}{3600\ \text{s}} \times \frac{\text{s}}{0.296\ \text{slug}} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times \frac{\text{lbm}\cdot{}^\circ\text{R}}{1\ \text{Btu}}$$

$$T_2 - T_1 = 0.995\ {}^\circ\text{R}$$

$T_2 - T_1$

This problem illustrates that:

- ✓ In general, the first law of thermodynamics and the Bernoulli equation are independent equations.
- ✓ For an incompressible, inviscid flow the internal thermal energy is only changed by a heat transfer process, and is independent of the fluid mechanics.

6-5 ENERGY GRADE LINE AND HYDRAULIC GRADE LINE

For steady, frictionless, incompressible flow along a streamline, we have shown that the first law of thermodynamics reduces to the Bernoulli equation. From Eq. 6.15 we conclude that there is no loss of mechanical energy in such a flow.

Often it is convenient to represent the mechanical energy level of a flow graphically. The energy equation in the form of Eq. 6.15 suggests such a representation. Dividing Eq. 6.15 by g, we obtain

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant} \tag{6.16}$$

Each term in Eq. 6.16 has dimensions of length, or "head" of flowing fluid. The individual terms are

$\frac{p}{\rho g}$, the head due to local static pressure
("pressure" energy per unit weight of the flowing fluid)

$\frac{V^2}{2g}$, the head due to local dynamic pressure
(kinetic energy per unit weight of flowing fluid)

z, the elevation head
(potential energy per unit weight of the flowing fluid)

H, the total head for the flow
(total mechanical energy per unit weight of the flowing fluid)

The *energy grade line* (EGL) represents the total head height. As shown by Eq. 6.16, the EGL height remains constant for frictionless flow when no work is done on or by the flowing liquid, although the individual static pressure, dynamic pressure, and elevation heads may vary. We recall from Section 6-3 that a pitot-static tube placed in the flow measures the stagnation pressure (static plus dynamic), and it will obviously be installed at the local height z of the flow; hence, the height of the liquid in a column attached to the tube will equal the sum of the three heads in Eq. 6.16. This height directly indicates the value of H, or the EGL.

The *hydraulic grade line* (HGL) height represents the sum of the elevation and static pressure heads, $z + p/\rho g$. In a static pressure tap attached to the flow conduit, liquid would rise to the HGL height. For open-channel flow, the HGL is at the liquid free surface.

The difference in heights between the EGL and the HGL represents the dynamic (velocity) head, $V^2/2g$. The relationship among the EGL, HGL, and velocity head is illustrated schematically in Fig. 6.6 for frictionless flow from a tank through a pipe with a reducer.

Static taps and total head tubes connected to manometers are shown schematically in Fig. 6.6. The *static taps* give readings corresponding to the *HGL height*. The *total head tubes* give readings corresponding to the *EGL height*.

The total head of the flow shown in Fig. 6.6 is obtained by applying Eq. 6.16 at point ①, the free surface in the large reservoir. There the velocity is negligible and the pressure is atmospheric (zero gage). Thus total head is equal to z_1. This defines the height of the energy grade line, which remains constant for this flow, since there is no friction or work.

The velocity head increases from zero to $V_2^2/2g$ as the liquid accelerates into the first section of constant-diameter tube. Hence, since the EGL height is constant the HGL must decrease in height. When the velocity becomes constant, the HGL height stays constant.

The velocity increases again in the reducer between sections ② and ③. As the velocity head increases, the HGL height drops. When the velocity becomes constant between sections ③ and ④, the HGL stays constant at a lower height.

At the free discharge at section ④, the static head is zero (gage). There the HGL height is equal to z_4. As shown, the velocity head is $V_4^2/2g$. The sum of the HGL height and velocity head equals the EGL height. (The static head is negative between sections ③ and ④ because the pipe centerline is above the HGL.)

The effects of friction on a flow will be discussed in detail in Chapter 8. The effect of friction is to convert mechanical energy to internal thermal energy. Thus *friction reduces the total head* of the flowing fluid, causing a gradual reduction in the EGL height.

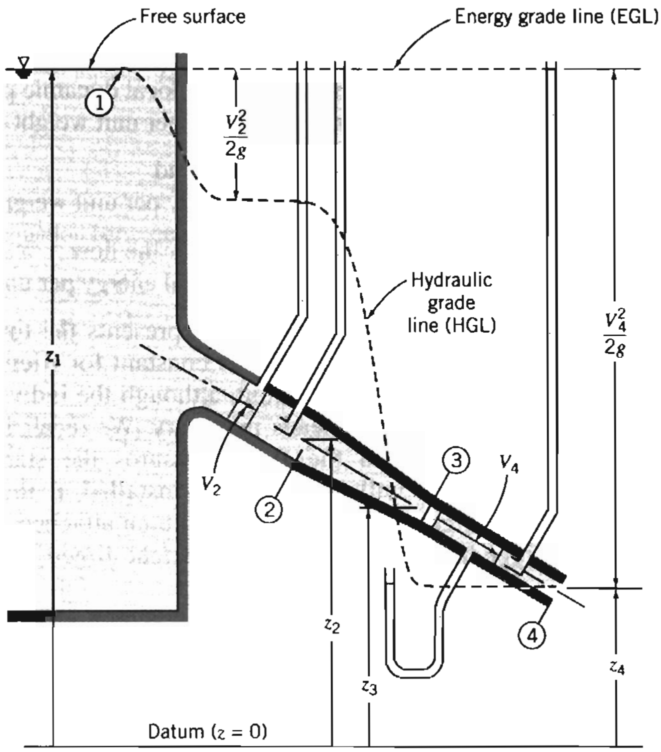


Fig. 6.6 Energy and hydraulic grade lines for frictionless flow.

Work addition to the fluid, for example as delivered by a pump, increases the EGL height. The effect of work interactions with a flow will be discussed in Chapters 8 and 10.

6-6 UNSTEADY BERNOULLI EQUATION—INTEGRATION OF EULER'S EQUATION ALONG A STREAMLINE (CD-ROM)

6-7 IRROTATIONAL FLOW (CD-ROM)

6-8 SUMMARY

In this chapter we have:

✓ Derived Euler's equations in vector form and in rectangular, cylindrical, and streamline coordinates.

✓ Obtained Bernoulli's equation by integrating Euler's equation along a steady-flow streamline, and discussed its restrictions. We have also seen how for a steady, incompressible flow through a stream tube the first law of thermodynamics reduces to the Bernoulli equation if certain restrictions apply.