Chapter 3
Integral Relations for a Control Volume

Motivation. In analyzing fluid motion, we might take one of two paths: (1) seeking to describe the detailed flow pattern at every point (x, y, z) in the field or (2) working with a finite region, making a balance of flow in versus flow out, and determining gross flow effects such as the force or torque on a body or the total energy exchange. The second is the "control-volume" method and is the subject of this chapter. The first is the "differential" approach and is developed in Chap. 4.

We first develop the concept of the control volume, in nearly the same manner as one does in a thermodynamics course, and we find the rate of change of an arbitrary gross fluid property, a result called the *Reynolds transport theorem*. We then apply this theorem, in sequence, to mass, linear momentum, angular momentum, and energy, thus deriving the four basic control-volume relations of fluid mechanics. There are many applications, of course. The chapter then ends with a special case of frictionless, shaft-work-free momentum and energy: the *Bernoulli equation*. The Bernoulli equation is a wonderful, historic relation, but it is extremely restrictive and should always be viewed with skepticism and care in applying it to a real (viscous) fluid motion.

3.1 Basic Physical Laws of Fluid Mechanics

It is time now to really get serious about flow problems. The fluid-statics applications of Chap. 2 were more like fun than work, at least in my opinion. Statics problems basically require only the density of the fluid and knowledge of the position of the free surface, but most flow problems require the analysis of an arbitrary state of variable fluid motion defined by the geometry, the boundary conditions, and the laws of mechanics. This chapter and the next two outline the three basic approaches to the analysis of arbitrary flow problems:

1. Control-volume, or large-scale, analysis (Chap. 3)
2. Differential, or small-scale, analysis (Chap. 4)
3. Experimental, or dimensional, analysis (Chap. 5)

The three approaches are roughly equal in importance, but control-volume analysis is "more equal," being the single most valuable tool to the engineer for flow analysis. It gives "engineering" answers, sometimes gross and crude but always useful. In principle, the differential approach of Chap. 4 can be used for any problem, but in practice the lack of mathematical tools and the inability of the digital computer to model small-scale processes make the differential approach rather limited. Similarly, although the dimensional analysis of Chap. 5 can be applied to any problem, the lack of time and money and generality often makes experimentation a limited approach. But a control-volume analysis takes about half an hour and gives useful results. Thus, in a trio of approaches, the control volume is best. Oddly enough, it is the newest of the three. Differential analysis began with Euler and Lagrange in the eighteenth century, and dimensional analysis was pioneered by Lord Rayleigh in the late nineteenth century, but the control volume, although proposed by Euler, was not developed on a rigorous basis as an analytical tool until the 1940s.

Systems versus Control Volumes

All the laws of mechanics are written for a *system*, which is defined as an arbitrary quantity of mass of fixed identity. Everything external to this system is denoted by the term *surroundings*, and the system is separated from its surroundings by its *boundaries*. The laws of mechanics then state what happens when there is an interaction between the system and its surroundings.

First, the system is a fixed quantity of mass, denoted by m. Thus the mass of the system is conserved and does not change.[1] This is a law of mechanics and has a very simple mathematical form, called *conservation of mass*:

$$m_{\text{syst}} = \text{const}$$

or

$$\frac{dm}{dt} = 0 \tag{3.1}$$

This is so obvious in solid-mechanics problems that we often forget about it. In fluid mechanics, we must pay a lot of attention to mass conservation, and it takes a little analysis to make it hold.

Second, if the surroundings exert a net force $\mathbf{F}$ on the system, Newton's second law states that the mass will begin to accelerate[2]

$$\mathbf{F} = m\mathbf{a} = m\,\frac{d\mathbf{V}}{dt} = \frac{d}{dt}\,(m\mathbf{V}) \tag{3.2}$$

In Eq. (2.12) we saw this relation applied to a differential element of viscous incompressible fluid. In fluid mechanics Newton's law is called the linear-momentum relation. Note that it is a vector law which implies the three scalar equations $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

Third, if the surroundings exert a net moment $\mathbf{M}$ about the center of mass of the system, there will be a rotation effect

$$\mathbf{M} = \frac{d\mathbf{H}}{dt} \tag{3.3}$$

where $\mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\ \delta m$ is the angular momentum of the system about its center of

[1] We are neglecting nuclear reactions, where mass can be changed to energy.
[2] We are neglecting relativistic effects, where Newton's law must be modified.

mass. Here we call Eq. (3.3) the angular-momentum relation. Note that it is also a vector equation implying three scalar equations such as $M_x = dH_x/dt$.

For an arbitrary mass and arbitrary moment, **H** is quite complicated and contains nine terms (see, e.g., Ref. 1, p. 285). In elementary dynamics we commonly treat only a rigid body rotating about a fixed x axis, for which Eq. (3.3) reduces to

$$M_x = I_x \frac{d}{dt}(\omega_x) \tag{3.4}$$

where ω_x is the angular velocity of the body and I_x is its mass moment of inertia about the x axis. Unfortunately, fluid systems are not rigid and rarely reduce to such a simple relation, as we shall see in Sec. 3.5.

Fourth, if heat dQ is added to the system or work dW is done by the system, the system energy dE must change according to the energy relation, or first law of thermodynamics,

$$dQ - dW = dE$$

or

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} \tag{3.5}$$

Like mass conservation, Eq. (3.1), this is a scalar relation having only a single component.

Finally, the second law of thermodynamics relates entropy change dS to heat added dQ and absolute temperature T:

$$dS \geq \frac{dQ}{T} \tag{3.6}$$

This is valid for a system and can be written in control-volume form, but there are almost no practical applications in fluid mechanics except to analyze flow-loss details (see Sec. 9.5).

All these laws involve thermodynamic properties, and thus we must supplement them with state relations $p = p(\rho, T)$ and $e = e(\rho, T)$ for the particular fluid being studied, as in Sec. 1.6.

The purpose of this chapter is to put our four basic laws into the control-volume form suitable for arbitrary regions in a flow:

1. Conservation of mass (Sec. 3.3)
2. The linear-momentum relation (Sec. 3.4)
3. The angular-momentum relation (Sec. 3.5)
4. The energy equation (Sec. 3.6)

Wherever necessary to complete the analysis we also introduce a state relation such as the perfect-gas law.

Equations (3.1) to (3.6) apply to either fluid or solid systems. They are ideal for solid mechanics, where we follow the same system forever because it represents the product we are designing and building. For example, we follow a beam as it deflects under load. We follow a piston as it oscillates. We follow a rocket system all the way to Mars.

But fluid systems do not demand this concentrated attention. It is rare that we wish to follow the ultimate path of a specific particle of fluid. Instead it is likely that the fluid forms the environment whose effect on our product we wish to know. For the three examples cited above, we wish to know the wind loads on the beam, the fluid pressures on the piston, and the drag and lift loads on the rocket. This requires that the basic laws be rewritten to apply to a specific *region* in the neighborhood of our product. In other words, where the fluid particles in the wind go after they leave the beam is of little interest to a beam designer. The user's point of view underlies the need for the control-volume analysis of this chapter.

Although thermodynamics is not at all the main topic of this book, it would be a shame if the student did not review at least the first law and the state relations, as discussed, e.g., in Refs. 6 and 7.

In analyzing a control volume, we convert the system laws to apply to a specific region which the system may occupy for only an instant. The system passes on, and other systems come along, but no matter. The basic laws are reformulated to apply to this local region called a control volume. All we need to know is the flow field in this region, and often simple assumptions will be accurate enough (e.g., uniform inlet and/or outlet flows). The flow conditions away from the control volume are then irrelevant. The technique for making such localized analyses is the subject of this chapter.

Volume and Mass Rate of Flow

All the analyses in this chapter involve evaluation of the volume flow Q or mass flow $\dot{m}$ passing through a surface (imaginary) defined in the flow.

Suppose that the surface S in Fig. 3.1*a* is a sort of (imaginary) wire mesh through which the fluid passes without resistance. How much volume of fluid passes through S in unit time? If, typically, $\mathbf{V}$ varies with position, we must integrate over the elemental surface dA in Fig. 3.1*a*. Also, typically $\mathbf{V}$ may pass through dA at an angle θ off the normal. Let $\mathbf{n}$ be defined as the unit vector normal to dA. Then the amount of fluid swept through dA in time dt is the volume of the slanted parallelopiped in Fig. 3.1*b*:

$$d\mathcal{V} = V\, dt\, dA \cos\theta = (\mathbf{V} \cdot \mathbf{n})\, dA\, dt$$

The integral of $d\mathcal{V}/dt$ is the total volume rate of flow Q through the surface S

$$Q = \int_S (\mathbf{V} \cdot \mathbf{n})\, dA = \int_S V_n\, dA \quad (3.7)$$

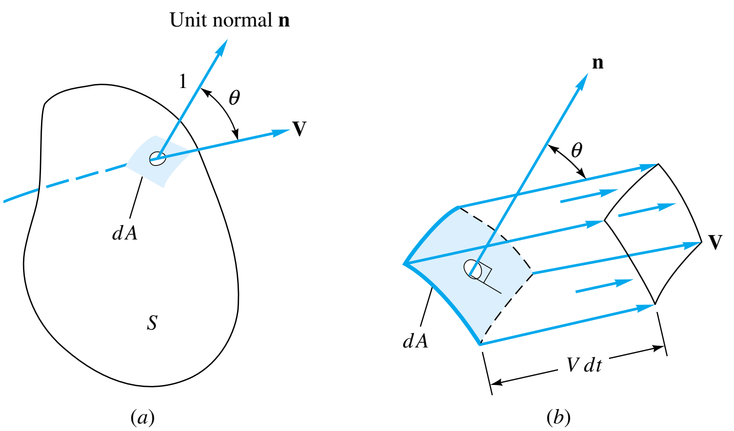


Fig. 3.1 Volume rate of flow through an arbitrary surface: (*a*) an elemental area dA on the surface; (*b*) the incremental volume swept through dA equals $V\, dt\, dA \cos\theta$.

We could replace $\mathbf{V} \cdot \mathbf{n}$ by its equivalent, V_n, the component of $\mathbf{V}$ normal to dA, but the use of the dot product allows Q to have a sign to distinguish between inflow and outflow. By convention throughout this book we consider $\mathbf{n}$ to be the *outward* normal unit vector. Therefore $\mathbf{V} \cdot \mathbf{n}$ denotes outflow if it is positive and inflow if negative. This will be an extremely useful housekeeping device when we are computing volume and mass flow in the basic control-volume relations.

Volume flow can be multiplied by density to obtain the mass flow $\dot{m}$. If density varies over the surface, it must be part of the surface integral

$$\dot{m} = \int_S \rho(\mathbf{V} \cdot \mathbf{n})\, dA = \int_S \rho V_n\, dA$$

If density is constant, it comes out of the integral and a direct proportionality results:

Constant density: $$\dot{m} = \rho Q$$

3.2 The Reynolds Transport Theorem

To convert a system analysis to a control-volume analysis, we must convert our mathematics to apply to a specific region rather than to individual masses. This conversion, called the *Reynolds transport theorem*, can be applied to all the basic laws. Examining the basic laws (3.1) to (3.3) and (3.5), we see that they are all concerned with the time derivative of fluid properties m, $\mathbf{V}$, $\mathbf{H}$, and E. Therefore what we need is to relate the time derivative of a system property to the rate of change of that property within a certain region.

The desired conversion formula differs slightly according to whether the control volume is fixed, moving, or deformable. Figure 3.2 illustrates these three cases. The fixed control volume in Fig. 3.2*a* encloses a stationary region of interest to a nozzle designer. The control surface is an abstract concept and does not hinder the flow in any way. It slices through the jet leaving the nozzle, circles around through the surrounding atmosphere, and slices through the flange bolts and the fluid within the nozzle. This particular control volume exposes the stresses in the flange bolts, which contribute to applied forces in the momentum analysis. In this sense the control volume resembles the *free-body* concept, which is applied to systems in solid-mechanics analyses.

Figure 3.2*b* illustrates a moving control volume. Here the ship is of interest, not the ocean, so that the control surface chases the ship at ship speed V. The control volume is of fixed volume, but the relative motion between water and ship must be considered.

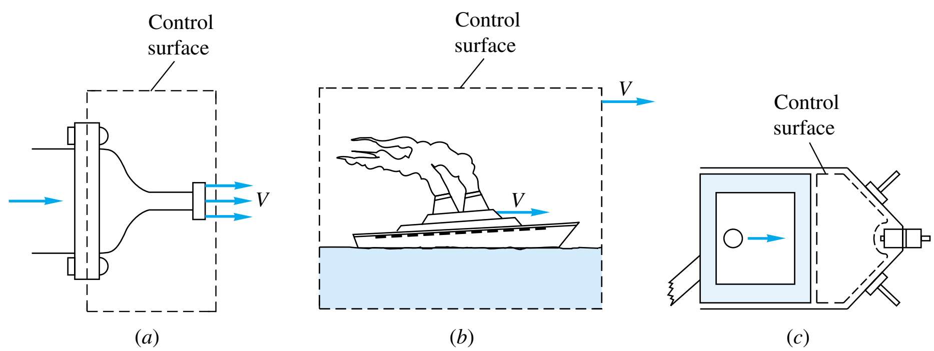


Fig. 3.2 Fixed, moving, and deformable control volumes: (*a*) fixed control volume for nozzle-stress analysis; (*b*) control volume moving at ship speed for drag-force analysis; (*c*) control volume deforming within cylinder for transient pressure-variation analysis.

If V is constant, this relative motion is a steady-flow pattern, which simplifies the analysis.[3] If V is variable, the relative motion is unsteady, so that the computed results are time-variable and certain terms enter the momentum analysis to reflect the noninertial frame of reference.

Figure 3.2*c* shows a deforming control volume. Varying relative motion at the boundaries becomes a factor, and the rate of change of shape of the control volume enters the analysis. We begin by deriving the fixed-control-volume case, and we consider the other cases as advanced topics.

One-Dimensional Fixed Control Volume

As a simple first example, consider a duct or streamtube with a nearly one-dimensional flow $V = V(x)$, as shown in Fig. 3.3. The selected control volume is a portion of the duct which happens to be filled exactly by system 2 at a particular instant t. At time $t + dt$, system 2 has begun to move out, and a sliver of system 1 has entered from the left. The shaded areas show an outflow sliver of volume $A_b V_b\, dt$ and an inflow volume $A_a V_a\, dt$.

Now let B be any property of the fluid (energy, momentum, etc.), and let $\beta = dB/dm$ be the *intensive* value or the amount of B per unit mass in any small portion of the fluid. The total amount of B in the control volume is thus

$$B_{CV} = \int_{CV} \beta\rho\, d\mathcal{V} \qquad \beta = \frac{dB}{dm} \tag{3.8}$$

[3]A *wind tunnel* uses a fixed model to simulate flow over a body moving through a fluid. A *tow tank* uses a moving model to simulate the same situation.

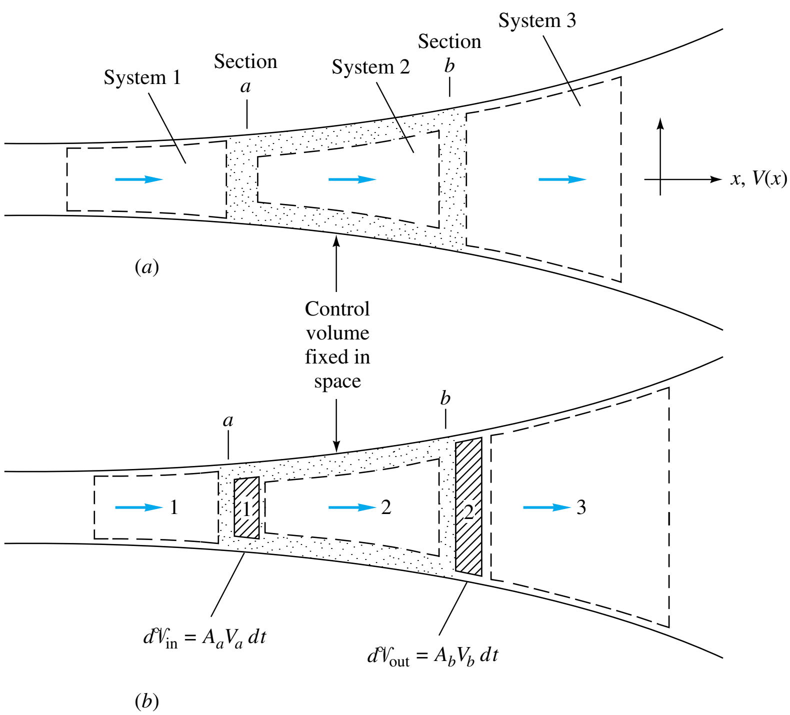


Fig. 3.3 Example of inflow and outflow as three systems pass through a control volume: (*a*) System 2 fills the control volume at time t; (*b*) at time $t + dt$ system 2 begins to leave and system 1 enters.

where $\rho\, d\mathcal{V}$ is a differential mass of the fluid. We want to relate the rate of change of B_{CV} to the rate of change of the amount of B in system 2 which happens to coincide with the control volume at time t. The time derivative of B_{CV} is defined by the calculus limit

$$\begin{aligned}\frac{d}{dt}(B_{CV}) &= \frac{1}{dt}B_{CV}(t+dt) - \frac{1}{dt}B_{CV}(t)\\ &= \frac{1}{dt}[B_2(t+dt) - (\beta\rho\, d\mathcal{V})_{out} + (\beta\rho d\mathcal{V})_{in}] - \frac{1}{dt}[B_2(t)]\\ &= \frac{1}{dt}[B_2(t+dt) - B_2(t)] - (\beta\rho AV)_{out} + (\beta\rho AV)_{in}\end{aligned}$$

The first term on the right is the rate of change of B within system 2 at the instant it occupies the control volume. By rearranging the last line of the above equation, we have the desired conversion formula relating changes in any property B of a local system to one-dimensional computations concerning a fixed control volume which instantaneously encloses the system.

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho\, d\mathcal{V}\right) + (\beta\rho AV)_{out} - (\beta\rho AV)_{in} \tag{3.9}$$

This is the one-dimensional Reynolds transport theorem for a fixed volume. The three terms on the right-hand side are, respectively,

1. The rate of change of B within the control volume
2. The flux of B passing out of the control surface
3. The flux of B passing into the control surface

If the flow pattern is steady, the first term vanishes. Equation (3.9) can readily be generalized to an arbitrary flow pattern, as follows.

Arbitrary Fixed Control Volume

Figure 3.4 shows a generalized fixed control volume with an arbitrary flow pattern passing through. The only additional complication is that there are variable slivers of inflow and outflow of fluid all about the control surface. In general, each differential area dA of surface will have a different velocity $\mathbf{V}$ making a different angle θ with the local normal to dA. Some elemental areas will have inflow volume $(VA\cos\theta)_{in}\, dt$, and others will have outflow volume $(VA\cos\theta)_{out}\, dt$, as seen in Fig. 3.4. Some surfaces might correspond to streamlines ($\theta = 90°$) or solid walls ($\mathbf{V} = 0$) with neither inflow nor outflow. Equation (3.9) generalizes to

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right) + \int_{CS} \beta\rho V\cos\theta\, dA_{out} - \int_{CS} \beta\rho V\cos\theta\, dA_{in} \tag{3.10}$$

This is the Reynolds transport theorem for an arbitrary fixed control volume. By letting the property B be mass, momentum, angular momentum, or energy, we can rewrite all the basic laws in control-volume form. Note that all three of the control-volume integrals are concerned with the intensive property β. Since the control volume is fixed in space, the elemental volumes $d\mathcal{V}$ do not vary with time, so that the time derivative of the volume integral vanishes unless either β or ρ varies with time (unsteady flow).

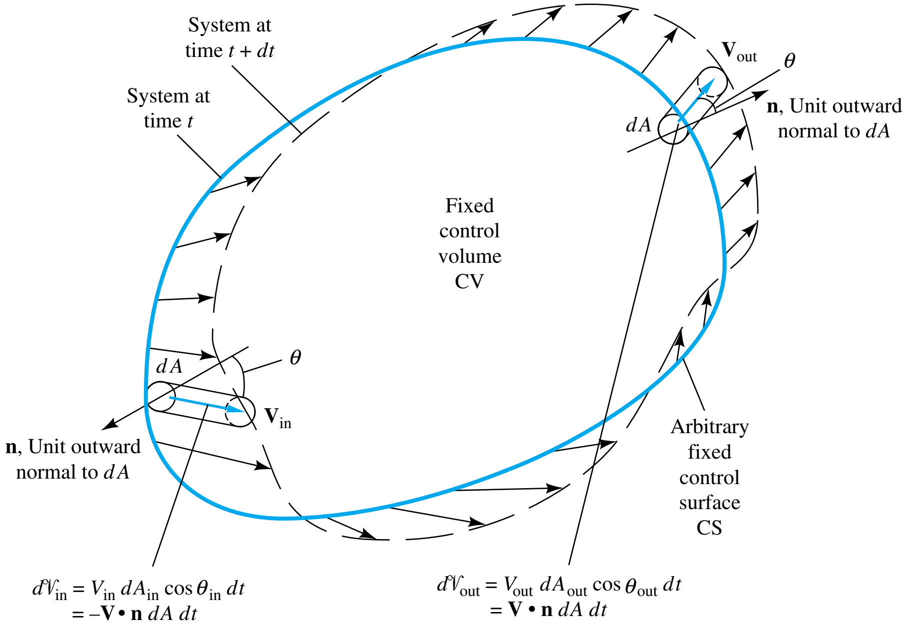


Fig. 3.4 Generalization of Fig. 3.3 to an arbitrary control volume with an arbitrary flow pattern.

Equation (3.10) expresses the basic formula that a system derivative equals the rate of change of B within the control volume plus the flux of B out of the control surface minus the flux of B into the control surface. The quantity B (or β) may be any vector or scalar property of the fluid. Two alternate forms are possible for the flux terms. First we may notice that $V \cos \theta$ is the component of V normal to the area element of the control surface. Thus we can write

$$\text{Flux terms} = \int_{CS} \beta\rho V_n\, dA_{out} - \int_{CS} \beta\rho V_n\, dA_{in} = \int_{CS} \beta\, d\dot{m}_{out} - \int_{CS} \beta\, d\dot{m}_{in} \qquad (3.11a)$$

where $d\dot{m} = \rho V_n\, dA$ is the differential mass flux through the surface. Form (3.11a) helps visualize what is being calculated.

A second alternate form offers elegance and compactness as advantages. If **n** is defined as the *outward* normal unit vector everywhere on the control surface, then $\mathbf{V} \cdot \mathbf{n} = V_n$ for outflow and $\mathbf{V} \cdot \mathbf{n} = -V_n$ for inflow. Therefore the flux terms can be represented by a single integral involving $\mathbf{V} \cdot \mathbf{n}$ which accounts for both positive outflow and negative inflow

$$\text{Flux terms} = \int_{CS} \beta\rho(\mathbf{V} \cdot \mathbf{n})\, dA \qquad (3.11b)$$

The compact form of the Reynolds transport theorem is thus

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho\, d\mathcal{V}\right) + \int_{CV} \beta\rho(\mathbf{V} \cdot \mathbf{n})\, dA \qquad (3.12)$$

This is beautiful but only occasionally useful, when the coordinate system is ideally suited to the control volume selected. Otherwise the computations are easier when the flux of B out is added and the flux of B in is subtracted, according to (3.10) or (3.11a).

The time-derivative term can be written in the equivalent form

$$\frac{d}{dt}\left(\int_{CV} \beta\rho \, d\mathcal{V}\right) = \int_{CV} \frac{\partial}{\partial t}(\beta\rho)\, d\mathcal{V} \tag{3.13}$$

for the fixed control volume since the volume elements do not vary.

Control Volume Moving at Constant Velocity

If the control volume is moving uniformly at velocity $\mathbf{V}_s$, as in Fig. 3.2*b*, an observer fixed to the control volume will see a relative velocity $\mathbf{V}_r$ of fluid crossing the control surface, defined by

$$\mathbf{V}_r = \mathbf{V} - \mathbf{V}_s \tag{3.14}$$

where $\mathbf{V}$ is the fluid velocity relative to the same coordinate system in which the control volume motion $\mathbf{V}_s$ is observed. Note that Eq. (3.14) is a vector subtraction. The flux terms will be proportional to $\mathbf{V}_r$, but the volume integral is unchanged because the control volume moves as a fixed shape without deforming. The Reynolds transport theorem for this case of a uniformly moving control volume is

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\left(\int_{CV} \beta\rho \, d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \tag{3.15}$$

which reduces to Eq. (3.12) if $\mathbf{V}_s \equiv 0$.

Control Volume of Constant Shape but Variable Velocity[4]

If the control volume moves with a velocity $\mathbf{V}_s(t)$ which retains its shape, then the volume elements do not change with time but the boundary relative velocity $\mathbf{V}_r = \mathbf{V}(\mathbf{r}, t) - \mathbf{V}_s(t)$ becomes a somewhat more complicated function. Equation (3.15) is unchanged in form, but the area integral may be more laborious to evaluate.

Arbitrarily Moving and Deformable Control Volume[5]

The most general situation is when the control volume is both moving and deforming arbitrarily, as illustrated in Fig. 3.5. The flux of volume across the control surface is again proportional to the relative normal velocity component $\mathbf{V}_r \cdot \mathbf{n}$, as in Eq. (3.15). However, since the control surface has a deformation, its velocity $\mathbf{V}_s = \mathbf{V}_s(\mathbf{r}, t)$, so that the relative velocity $\mathbf{V}_r = \mathbf{V}(\mathbf{r}, t) - \mathbf{V}_s(\mathbf{r}, t)$ is or can be a complicated function, even though the flux integral is the same as in Eq. (3.15). Meanwhile, the volume integral in Eq. (3.15) must allow the volume elements to distort with time. Thus the time derivative must be applied *after* integration. For the deforming control volume, then, the transport theorem takes the form

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\left(\int_{CV} \beta\rho \, d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \tag{3.16}$$

This is the most general case, which we can compare with the equivalent form for a fixed control volume

[4]This section may be omitted without loss of continuity.

[5]This section may be omitted without loss of continuity.

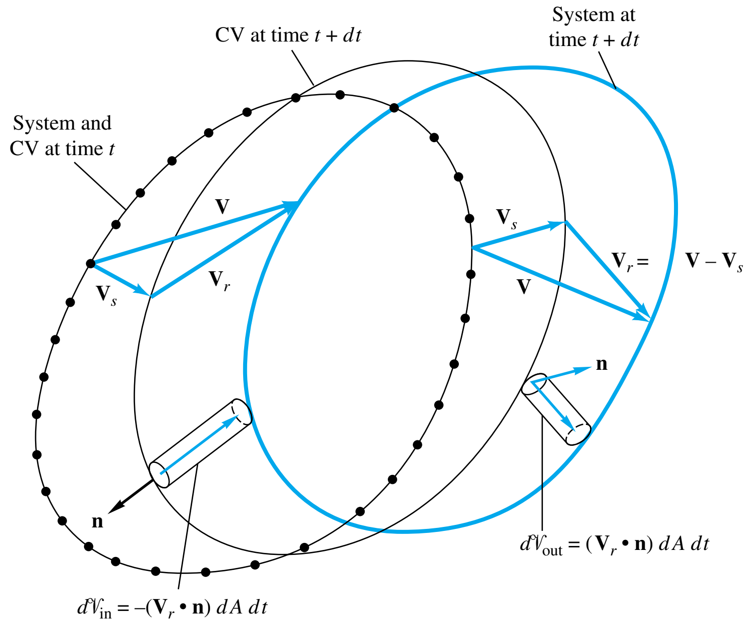


Fig. 3.5 Relative-velocity effects between a system and a control volume when both move and deform. The system boundaries move at velocity **V**, and the control surface moves at velocity $\mathbf{V}_s$.

$$\frac{d}{dt}(B_{\text{syst}}) = \int_{\text{CV}} \frac{\partial}{\partial t}(\beta\rho)\, d\mathcal{V} + \int_{\text{CS}} \beta\rho(\mathbf{V}\cdot\mathbf{n})\, dA \tag{3.17}$$

The moving and deforming control volume, Eq. (3.16), contains only two complications: (1) The time derivative of the first integral on the right must be taken outside, and (2) the second integral involves the *relative* velocity $\mathbf{V}_r$ between the fluid system and the control surface. These differences and mathematical subtleties are best shown by examples.

One-Dimensional Flux-Term Approximations

In many applications, the flow crosses the boundaries of the control surface only at certain simplified inlets and exits which are approximately *one-dimensional*; i.e., the flow properties are nearly uniform over the cross section of the inlet or exit. Then the double-integral flux terms required in Eq. (3.16) reduce to a simple sum of positive (exit) and negative (inlet) product terms involving the flow properties at each cross section

$$\int_{\text{CS}} \beta\rho(\mathbf{V}_r\cdot\mathbf{n})\, dA = \sum(\beta_i\rho_i V_{ri}A_i)_{\text{out}} - \sum(\beta_i\rho_i V_{ri}A_i)_{\text{in}} \tag{3.18}$$

An example of this situation is shown in Fig. 3.6. There are inlet flows at sections 1 and 4 and outflows at sections 2, 3, and 5. For this particular problem Eq. (3.18) would be

$$\int_{\text{CS}} \beta\rho(\mathbf{V}_r\cdot\mathbf{n})\, dA = \beta_2\rho_2V_{r2}A_2 + \beta_3\rho_3V_{r3}A_3 + \beta_5\rho_5V_{r5}A_5 - \beta_1\rho_1V_{r1}A_1 - \beta_4\rho_4V_{r4}A_4 \tag{3.19}$$

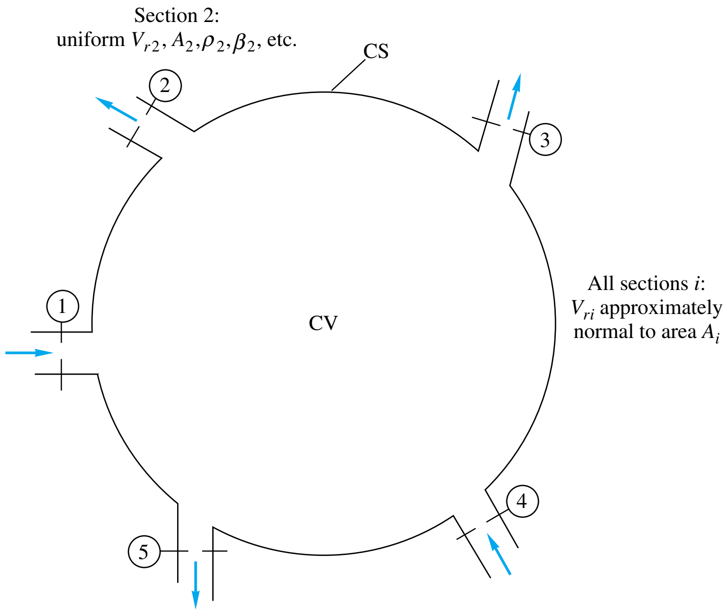
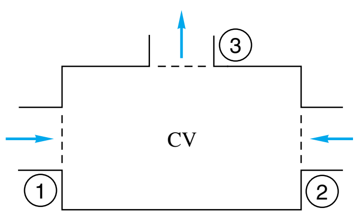


Fig. 3.6 A control volume with simplified one-dimensional inlets and exits.

with no contribution from any other portion of the control surface because there is no flow across the boundary.

EXAMPLE 3.1

A fixed control volume has three one-dimensional boundary sections, as shown in Fig. E3.1. The flow within the control volume is steady. The flow properties at each section are tabulated below. Find the rate of change of energy of the system which occupies the control volume at this instant.



E3.1

Section	Type	ρ, kg/m^3	V, m/s	A, m^2	e, J/kg
1	Inlet	800	5.0	2.0	300
2	Inlet	800	8.0	3.0	100
3	Outlet	800	17.0	2.0	150

Solution

The property under study here is energy, and so $B = E$ and $\beta = dE/dm = e$, the energy per unit mass. Since the control volume is fixed, Eq. (3.17) applies:

$$\left(\frac{dE}{dt}\right)_{\text{syst}} = \int_{\text{CV}} \frac{\partial}{\partial t}(e\rho)\, d\mathcal{V} + \int_{\text{CS}} e\rho(\mathbf{V}\cdot\mathbf{n})\, dA$$

The flow within is steady, so that $\partial(e\rho)/\partial t \equiv 0$ and the volume integral vanishes. The area integral consists of two inlet sections and one outlet section, as given in the table

$$\left(\frac{dE}{dt}\right)_{\text{syst}} = -e_1\rho_1A_1V_1 - e_2\rho_2A_2V_2 + e_3\rho_3A_3V_3$$

Introducing the numerical values from the table, we have

$$\left(\frac{dE}{dt}\right)_{syst} = -(300\text{ J/kg})(800\text{ kg/m}^3)(2\text{ m}^2)(5\text{ m/s}) - 100(800)(3)(8) + 150(800)(2)(17)$$

$$= (-2{,}400{,}000 - 1{,}920{,}000 + 4{,}080{,}000)\text{ J/s}$$

$$= -240{,}000\text{ J/s} = -0.24\text{ MJ/s} \qquad \textit{Ans.}$$

Thus the system is losing energy at the rate of 0.24 MJ/s = 0.24 MW. Since we have accounted for all fluid energy crossing the boundary, we conclude from the first law that there must be heat loss through the control surface or the system must be doing work on the environment through some device not shown. Notice that the use of SI units leads to a consistent result in joules per second without any conversion factors. We promised in Chap. 1 that this would be the case.

Note: This problem involves energy, but suppose we check the balance of mass also. Then B = mass m, and $B = dm/dm$ = unity. Again the volume integral vanishes for steady flow, and Eq. (3.17) reduces to

$$\left(\frac{dm}{dt}\right)_{syst} = \int_{CS} \rho(\mathbf{V}\cdot\mathbf{n})\,dA = -\rho_1 A_1 V_1 - \rho_2 A_2 V_2 + \rho_3 A_3 V_3$$

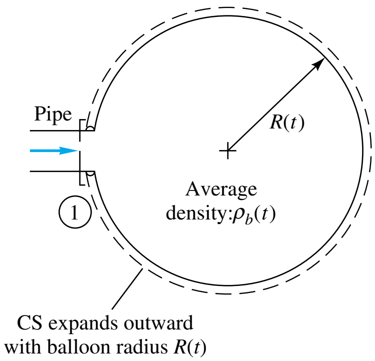
$$= -(800\text{ kg/m}^3)(2\text{ m}^2)(5\text{ m/s}) - 800(3)(8) + 800(17)(2)$$

$$= (-8000 - 19{,}200 + 27{,}200)\text{ kg/s} = 0\text{ kg/s}$$

Thus the system mass does not change, which correctly expresses the law of conservation of system mass, Eq. (3.1).

EXAMPLE 3.2

The balloon in Fig. E3.2 is being filled through section 1, where the area is A_1, velocity is V_1, and fluid density is ρ_1. The average density within the balloon is $\rho_b(t)$. Find an expression for the rate of change of system mass within the balloon at this instant.



E3.2

Solution

It is convenient to define a deformable control surface just outside the balloon, expanding at the same rate $R(t)$. Equation (3.16) applies with $V_r = 0$ on the balloon surface and $V_r = V_1$ at the pipe entrance. For mass change, we take $B = m$ and $\beta = dm/dm = 1$. Equation (3.16) becomes

$$\left(\frac{dm}{dt}\right)_{syst} = \frac{d}{dt}\left(\int_{CS} \rho\, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V_r}\cdot\mathbf{n})\,dA$$

Mass flux occurs only at the inlet, so that the control-surface integral reduces to the single negative term $-\rho_1 A_1 V_1$. The fluid mass within the control volume is approximately the average density times the volume of a sphere. The equation thus becomes

$$\left(\frac{dm}{dt}\right)_{syst} = \frac{d}{dt}\left(\rho_b \frac{4}{3}\pi R^3\right) - \rho_1 A_1 V_1 \qquad \textit{Ans.}$$

This is the desired result for the system mass rate of change. Actually, by the conservation law

(3.1), this change must be zero. Thus the balloon density and radius are related to the inlet mass flux by

$$\frac{d}{dt}(\rho_b R^3) = \frac{3}{4\pi}\rho_1 A_1 V_1$$

This is a first-order differential equation which could form part of an engineering analysis of balloon inflation. It cannot be solved without further use of mechanics and thermodynamics to relate the four unknowns ρ_b, ρ_1, V_1, and R. The pressure and temperature and the elastic properties of the balloon would also have to be brought into the analysis.

For advanced study, many more details of the analysis of deformable control volumes can be found in Hansen [4] and Potter and Foss [5].

3.3 Conservation of Mass

The Reynolds transport theorem, Eq. (3.16) or (3.17), establishes a relation between system rates of change and control-volume surface and volume integrals. But system derivatives are related to the basic laws of mechanics, Eqs. (3.1) to (3.5). Eliminating system derivatives between the two gives the control-volume, or *integral*, forms of the laws of mechanics of fluids. The dummy variable B becomes, respectively, mass, linear momentum, angular momentum, and energy.

For conservation of mass, as discussed in Examples 3.1 and 3.2, $B = m$ and $\beta = dm/dm = 1$. Equation (3.1) becomes

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = 0 = \frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n}) \, dA \qquad (3.20)$$

This is the integral mass-conservation law for a deformable control volume. For a fixed control volume, we have

$$\int_{CV} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{CS} \rho(\mathbf{V} \cdot \mathbf{n}) \, dA = 0 \qquad (3.21)$$

If the control volume has only a number of one-dimensional inlets and outlets, we can write

$$\int_{CV} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{\text{out}} - \sum_i (\rho_i A_i V_i)_{\text{in}} = 0 \qquad (3.22)$$

Other special cases occur. Suppose that the flow within the control volume is steady; then $\partial\rho/\partial t \equiv 0$, and Eq. (3.21) reduces to

$$\int_{CS} \rho(\mathbf{V} \cdot \mathbf{n}) \, dA = 0 \qquad (3.23)$$

This states that in steady flow the mass flows entering and leaving the control volume must balance exactly.[6] If, further, the inlets and outlets are one-dimensional, we have

[6]Throughout this section we are neglecting *sources* or *sinks* of mass which might be embedded in the control volume. Equations (3.20) and (3.21) can readily be modified to add source and sink terms, but this is rarely necessary.

for steady flow

$$\sum_i (\rho_i A_i V_i)_{\text{in}} = \sum_i (\rho_i A_i V_i)_{\text{out}} \tag{3.24}$$

This simple approximation is widely used in engineering analyses. For example, referring to Fig. 3.6, we see that if the flow in that control volume is steady, the three outlet mass fluxes balance the two inlets:

$$\text{Outflow} = \text{inflow}$$
$$\rho_2 A_2 V_2 + \rho_3 A_3 V_3 + \rho_5 A_5 V_5 = \rho_1 A_1 V_1 + \rho_4 A_4 V_4 \tag{3.25}$$

The quantity ρAV is called the *mass flow* $\dot{m}$ passing through the one-dimensional cross section and has consistent units of kilograms per second (or slugs per second) for SI (or BG) units. Equation (3.25) can be rewritten in the short form

$$\dot{m}_2 + \dot{m}_3 + \dot{m}_5 = \dot{m}_1 + \dot{m}_4 \tag{3.26}$$

and, in general, the steady-flow–mass-conservation relation (3.23) can be written as

$$\sum_i (\dot{m}_i)_{\text{out}} = \sum_i (\dot{m}_i)_{\text{in}} \tag{3.27}$$

If the inlets and outlets are not one-dimensional, one has to compute $\dot{m}$ by integration over the section

$$\dot{m}_{\text{cs}} = \int_{\text{cs}} \rho(\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.28}$$

where "cs" stands for cross section. An illustration of this is given in Example 3.4.

Incompressible Flow

Still further simplification is possible if the fluid is incompressible, which we may define as having density variations which are negligible in the mass-conservation requirement.[7] As we saw in Chap. 1, all liquids are nearly incompressible, and gas flows can *behave* as if they were incompressible, particularly if the gas velocity is less than about 30 percent of the speed of sound of the gas.

Again consider the fixed control volume. If the fluid is nearly incompressible, $\partial\rho/\partial t$ is negligible and the volume integral in Eq. (3.21) may be neglected, after which the density can be slipped outside the surface integral and divided out since it is nonzero. The result is a conservation law for incompressible flows, whether steady or unsteady:

$$\int_{\text{CS}} (\mathbf{V} \cdot \mathbf{n})\, dA = 0 \tag{3.29}$$

If the inlets and outlets are one-dimensional, we have

$$\sum_i (V_i A_i)_{\text{out}} = \sum_i (V_i A_i)_{\text{in}} \tag{3.30}$$

or

$$\sum Q_{\text{out}} = \sum Q_{\text{in}}$$

where $Q_i = V_i A_i$ is called the *volume flow* passing through the given cross section.

[7] Be warned that there is subjectivity in specifying incompressibility. Oceanographers consider a 0.1 percent density variation very significant, while aerodynamicists often neglect density variations in highly compressible, even hypersonic, gas flows. Your task is to justify the incompressible approximation when you make it.

Again, if consistent units are used, $Q = VA$ will have units of cubic meters per second (SI) or cubic feet per second (BG). If the cross section is not one-dimensional, we have to integrate

$$Q_{CS} = \int_{CS} (\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.31}$$

Equation (3.31) allows us to define an *average velocity* V_{av} which, when multiplied by the section area, gives the correct volume flow

$$V_{av} = \frac{Q}{A} = \frac{1}{A} \int (\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.32}$$

This could be called the *volume-average velocity*. If the density varies across the section, we can define an average density in the same manner:

$$\rho_{av} = \frac{1}{A} \int \rho\, dA \tag{3.33}$$

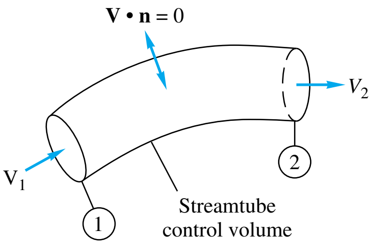
But the mass flow would contain the product of density and velocity, and the average product $(\rho V)_{av}$ would in general have a different value from the product of the averages

$$(\rho V)_{av} = \frac{1}{A} \int \rho (\mathbf{V} \cdot \mathbf{n})\, dA \approx \rho_{av} V_{av} \tag{3.34}$$

We illustrate average velocity in Example 3.4. We can often neglect the difference or, if necessary, use a correction factor between mass average and volume average.

EXAMPLE 3.3

Write the conservation-of-mass relation for steady flow through a streamtube (flow everywhere parallel to the walls) with a single one-dimensional exit 1 and inlet 2 (Fig. E3.3).



E3.3

Solution

For steady flow Eq. (3.24) applies with the single inlet and exit

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const}$$

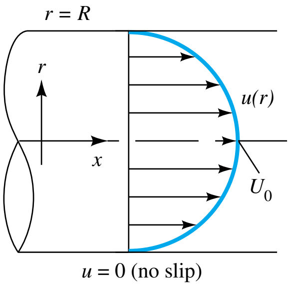
Thus, in a streamtube in steady flow, the mass flow is constant across every section of the tube. If the density is constant, then

$$Q = A_1 V_1 = A_2 V_2 = \text{const} \qquad \text{or} \qquad V_2 = \frac{A_1}{A_2} V_1$$

The volume flow is constant in the tube in steady incompressible flow, and the velocity increases as the section area decreases. This relation was derived by Leonardo da Vinci in 1500.

EXAMPLE 3.4

For steady viscous flow through a circular tube (Fig. E3.4), the axial velocity profile is given approximately by



E3.4

$$u = U_0\left(1 - \frac{r}{R}\right)^m$$

so that u varies from zero at the wall ($r = R$), or no slip, up to a maximum $u = U_0$ at the centerline $r = 0$. For highly viscous (laminar) flow $m \approx \frac{1}{2}$, while for less viscous (turbulent) flow $m \approx \frac{1}{7}$. Compute the average velocity if the density is constant.

Solution

The average velocity is defined by Eq. (3.32). Here $\mathbf{V} = \mathbf{i}u$ and $\mathbf{n} = \mathbf{i}$, and thus $\mathbf{V} \cdot \mathbf{n} = u$. Since the flow is symmetric, the differential area can be taken as a circular strip $dA = 2\,\pi r\,dr$. Equation (3.32) becomes

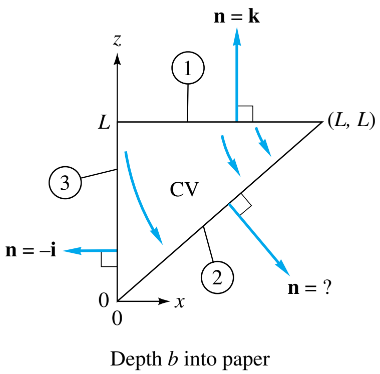
$$V_{av} = \frac{1}{A}\int u\,dA = \frac{1}{\pi R^2}\int_0^R U_0\left(1 - \frac{r}{R}\right)^m 2\pi r\,dr$$

or

$$V_{av} = U_0\frac{2}{(1 + m)(2 + m)} \qquad \textit{Ans.}$$

For the laminar-flow approximation, $m \approx \frac{1}{2}$ and $V_{av} \approx 0.53U_0$. (The exact laminar theory in Chap. 6 gives $V_{av} = 0.50U_0$.) For turbulent flow, $m \approx \frac{1}{7}$ and $V_{av} \approx 0.82U_0$. (There is no exact turbulent theory, and so we accept this approximation.) The turbulent velocity profile is more uniform across the section, and thus the average velocity is only slightly less than maximum.

EXAMPLE 3.5



E3.5

Consider the constant-density velocity field

$$u = \frac{V_0 x}{L} \qquad \upsilon = 0 \qquad w = -\frac{V_0 z}{L}$$

similar to Example 1.10. Use the triangular control volume in Fig. E3.5, bounded by (0, 0), (L, L), and (0, L), with depth b into the paper. Compute the volume flow through sections 1, 2, and 3, and compare to see whether mass is conserved.

Solution

The velocity field everywhere has the form $\mathbf{V} = \mathbf{i}u + \mathbf{k}w$. This must be evaluated along each section. We save section 2 until last because it looks tricky. Section 1 is the plane $z = L$ with depth b. The unit outward normal is $\mathbf{n} = \mathbf{k}$, as shown. The differential area is a strip of depth b varying with x: $dA = b\,dx$. The normal velocity is

$$(\mathbf{V} \cdot \mathbf{n})_1 = (\mathbf{i}u + \mathbf{k}w) \cdot \mathbf{k} = w|_1 = -\frac{V_0 z}{L}\bigg|_{z=L} = -V_0$$

The volume flow through section 1 is thus, from Eq. (3.31),

$$Q_1 = \int_1 (\mathbf{V} \cdot \mathbf{n})\,dA = \int_0^L (-V_0)b\,dx = -V_0 bL \qquad \textit{Ans. 1}$$

Since this is negative, section 1 is a net inflow. Check the units: V_0bL is a velocity times an area; OK.

Section 3 is the plane $x = 0$ with depth b. The unit normal is $\mathbf{n} = -\mathbf{i}$, as shown, and $dA = b\,dz$. The normal velocity is

$$(\mathbf{V} \cdot \mathbf{n})_3 = (\mathbf{i}u + \mathbf{k}w) \cdot (-\mathbf{i}) = -u\big|_3 = -\frac{V_0 x}{L}\bigg|_{s=0} = 0 \qquad \textit{Ans. 3}$$

Thus $V_n \equiv 0$ all along section 3; hence $Q_3 = 0$.

Finally, section 2 is the plane $x = z$ with depth b. The normal direction is to the right $\mathbf{i}$ and down $-\mathbf{k}$ but must have *unit* value; thus $\mathbf{n} = (1/\sqrt{2})(\mathbf{i} - \mathbf{k})$. The differential area is either $dA = \sqrt{2}b\,dx$ or $dA = \sqrt{2}b\,dz$. The normal velocity is

$$(\mathbf{V} \cdot \mathbf{n})_2 = (\mathbf{i}u + \mathbf{k}w) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}) = \frac{1}{\sqrt{2}}(u - w)_2$$

$$= \frac{1}{\sqrt{2}}\left[V_0 \frac{x}{L} - \left(-V_0 \frac{z}{L}\right)\right]_{x=z} = \frac{\sqrt{2}V_0 x}{L} \quad \text{or} \quad \frac{\sqrt{2}V_0 z}{L}$$

Then the volume flow through section 2 is

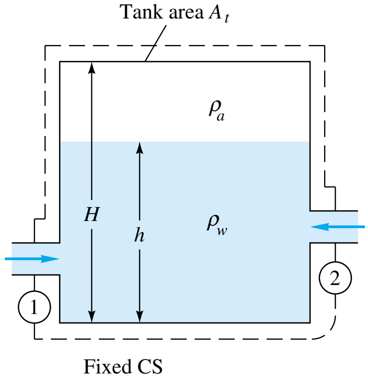
$$Q_2 = \int_2 (\mathbf{V} \cdot \mathbf{n})\,dA = \int_0^L \frac{\sqrt{2}V_0 x}{L}(\sqrt{2}b\,dx) = V_0 bL \qquad \textit{Ans. 2}$$

This answer is positive, indicating an outflow. These are the desired results. We should note that the volume flow is zero through the front and back triangular faces of the prismatic control volume because $V_n = v = 0$ on those faces.

The sum of the three volume flows is

$$Q_1 + Q_2 + Q_3 = -V_0 bL + V_0 bL + 0 = 0$$

Mass is conserved in this constant-density flow, and there are no net sources or sinks within the control volume. This is a very realistic flow, as described in Example 1.10



E3.6

EXAMPLE 3.6

The tank in Fig. E3.6 is being filled with water by two one-dimensional inlets. Air is trapped at the top of the tank. The water height is h. (*a*) Find an expression for the change in water height dh/dt. (*b*) Compute dh/dt if $D_1 = 1$ in, $D_2 = 3$ in, $V_1 = 3$ ft/s, $V_2 = 2$ ft/s, and $A_t = 2$ ft^2, assuming water at 20°C.

Solution

Part (a)

A suggested control volume encircles the tank and cuts through the two inlets. The flow within is unsteady, and Eq. (3.22) applies with no outlets and two inlets:

$$\frac{d}{dt}\left(\int_{CV} \rho\, d\mathcal{V}\right) - \rho_1 A_1 V_1 - \rho_2 A_2 V_2 = 0 \qquad (1)$$

Now if A_t is the tank cross-sectional area, the unsteady term can be evaluated as follows:

$$\frac{d}{dt}\left(\int_{CV} \rho\, d\mathcal{V}\right) = \frac{d}{dt}(\rho_w A_t h) + \frac{d}{dt}[\rho_a A_t (H - h)] = \rho_w A_t \frac{dh}{dt} \qquad (2)$$

The ρ_a term vanishes because it is the rate of change of air mass and is zero because the air is trapped at the top. Substituting (2) into (1), we find the change of water height

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A_t} \qquad \textit{Ans. (a)}$$

For water, $\rho_1 = \rho_2 = \rho_w$, and this result reduces to

$$\frac{dh}{dt} = \frac{A_1 V_1 + A_2 V_2}{A_t} = \frac{Q_1 + Q_2}{A_t} \qquad (3)$$

Part (b) The two inlet volume flows are

$$Q_1 = A_1 V_1 = \tfrac{1}{4}\pi(\tfrac{1}{12}\text{ ft})^2(3\text{ ft/s}) = 0.016\text{ ft}^3/\text{s}$$

$$Q_2 = A_2 V_2 = \tfrac{1}{4}\pi(\tfrac{3}{12}\text{ ft})^2(2\text{ ft/s}) = 0.098\text{ ft}^3/\text{s}$$

Then, from Eq. (3),

$$\frac{dh}{dt} = \frac{(0.016 + 0.098)\text{ ft}^3/\text{s}}{2\text{ ft}^2} = 0.057\text{ ft/s} \qquad \textit{Ans. (b)}$$

Suggestion: Repeat this problem with the top of the tank open.

An illustration of a mass balance with a deforming control volume has already been given in Example 3.2.

The control-volume mass relations, Eq. (3.20) or (3.21), are fundamental to all fluid-flow analyses. They involve only velocity and density. Vector directions are of no consequence except to determine the normal velocity at the surface and hence whether the flow is *in* or *out*. Although your specific analysis may concern forces or moments or energy, you must always make sure that mass is balanced as part of the analysis; otherwise the results will be unrealistic and probably rotten. We shall see in the examples which follow how mass conservation is constantly checked in performing an analysis of other fluid properties.

3.4 The Linear Momentum Equation

In Newton's law, Eq. (3.2), the property being differentiated is the linear momentum $m\mathbf{V}$. Therefore our dummy variable is $\mathbf{B} = m\mathbf{V}$ and $\boldsymbol{\beta} = d\mathbf{B}/dm = \mathbf{V}$, and application of the Reynolds transport theorem gives the linear-momentum relation for a deformable control volume

$$\frac{d}{dt}(m\mathbf{V})_{\text{syst}} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{\text{CV}} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{\text{CS}} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \qquad (3.35)$$

The following points concerning this relation should be strongly emphasized:

1. The term $\mathbf{V}$ is the fluid velocity relative to an *inertial* (nonaccelerating) coordinate system; otherwise Newton's law must be modified to include noninertial relative-acceleration terms (see the end of this section).
2. The term $\Sigma\, \mathbf{F}$ is the *vector* sum of all forces acting on the control-volume material considered as a free body; i.e., it includes surface forces on all fluids and

solids cut by the control surface plus all body forces (gravity and electromagnetic) acting on the masses within the control volume.

3. The entire equation is a vector relation; both the integrals are vectors due to the term **V** in the integrands. The equation thus has three components. If we want only, say, the x component, the equation reduces to

$$\sum F_x = \frac{d}{dt}\left(\int_{CV} u\rho \, d\mathcal{V}\right) + \int_{CS} u\rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA \tag{3.36}$$

and similarly, ΣF_y and ΣF_z would involve v and w, respectively. Failure to account for the vector nature of the linear-momentum relation (3.35) is probably the greatest source of student error in control-volume analyses.

For a fixed control volume, the relative velocity $\mathbf{V}_r \equiv \mathbf{V}$, and

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \tag{3.37}$$

Again we stress that this is a vector relation and that **V** must be an inertial-frame velocity. Most of the momentum analyses in this text are concerned with Eq. (3.37).

One-Dimensional Momentum Flux

By analogy with the term *mass flow* used in Eq. (3.28), the surface integral in Eq. (3.37) is called the *momentum-flux term.* If we denote momentum by **M**, then

$$\dot{\mathbf{M}}_{CS} = \int_{sec} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \tag{3.38}$$

Because of the dot product, the result will be negative for inlet momentum flux and positive for outlet flux. If the cross section is one-dimensional, **V** and ρ are uniform over the area and the integrated result is

$$\dot{\mathbf{M}}_{seci} = \mathbf{V}_i(\rho_i V_{ni} A_i) = \dot{m}_i \mathbf{V}_i \tag{3.39}$$

for outlet flux and $-\dot{m}_i\mathbf{V}_i$ for inlet flux. Thus if the control volume has only one-dimensional inlets and outlets, Eq. (3.37) reduces to

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \sum(\dot{m}_i \mathbf{V}_i)_{out} - \sum(\dot{m}_i \mathbf{V}_i)_{in} \tag{3.40}$$

This is a commonly used approximation in engineering analyses. It is crucial to realize that we are dealing with vector sums. Equation (3.40) states that the net vector force on a fixed control volume equals the rate of change of vector momentum within the control volume plus the vector sum of outlet momentum fluxes minus the vector sum of inlet fluxes.

Net Pressure Force on a Closed Control Surface

Generally speaking, the surface forces on a control volume are due to (1) forces exposed by cutting through solid bodies which protrude through the surface and (2) forces due to pressure and viscous stresses of the surrounding fluid. The computation of pressure force is relatively simple, as shown in Fig. 3.7. Recall from Chap. 2 that the external pressure force on a surface is normal to the surface and *inward*. Since the unit vector **n** is defined as *outward*, one way to write the pressure force is

$$\mathbf{F}_{press} = \int_{CS} p(-\mathbf{n}) \, dA \tag{3.41}$$

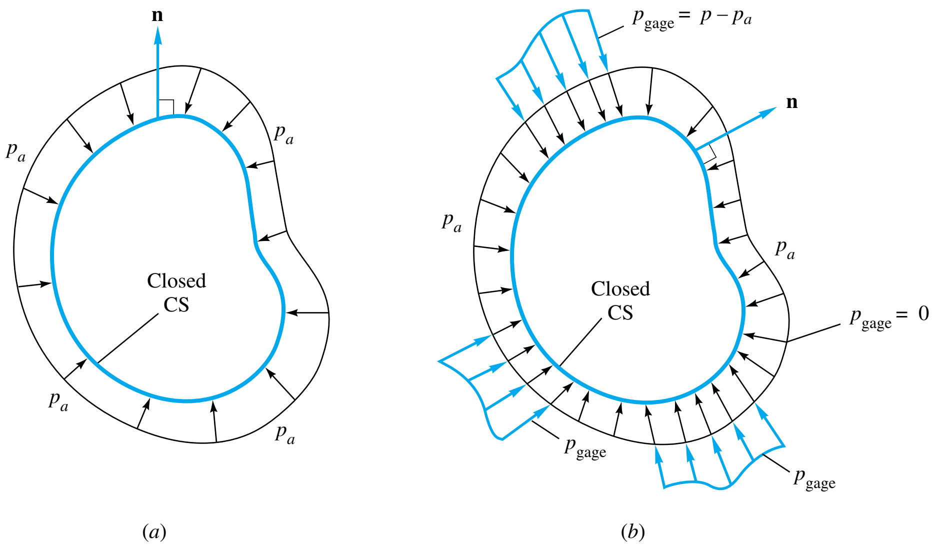


Fig. 3.7 Pressure-force computation by subtracting a uniform distribution: (*a*) uniform pressure, $\mathbf{F} = -p_a \int \mathbf{n}\, dA \equiv 0$; (*b*) nonuniform pressure, $\mathbf{F} = -\int (p - p_a)\mathbf{n}\, dA$.

Now if the pressure has a uniform value p_a all around the surface, as in Fig. 3.7*a*, the net pressure force is zero

$$\mathbf{F}_{\text{UP}} = \int p_a(-\mathbf{n})\, dA = -p_a \int \mathbf{n}\, dA \equiv 0 \tag{3.42}$$

where the subscript UP stands for uniform pressure. This result is *independent of the shape of the surface*[8] as long as the surface is closed and all our control volumes are closed. Thus a seemingly complicated pressure-force problem can be simplified by subtracting any convenient uniform pressure p_a and working only with the pieces of gage pressure which remain, as illustrated in Fig. 3.7*b*. Thus Eq. (3.41) is entirely equivalent to

$$\mathbf{F}_{\text{press}} = \int_{\text{CS}} (p - p_a)(-\mathbf{n})\, dA = \int_{\text{CS}} p_{\text{gage}}(-\mathbf{n})\, dA$$

This trick can mean quite a saving in computation.

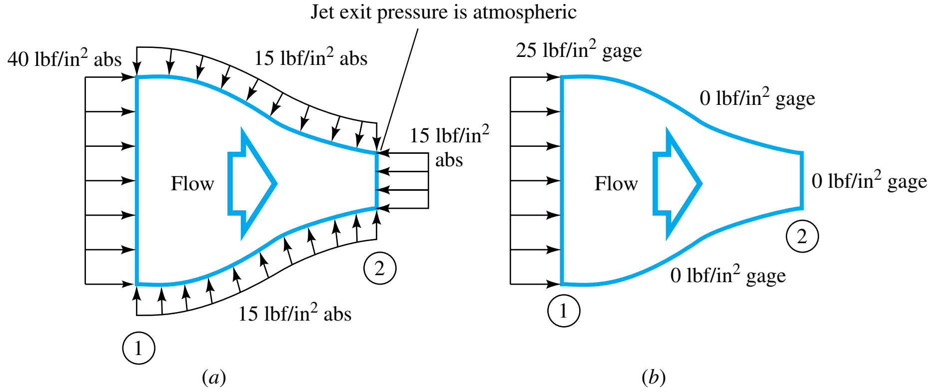
EXAMPLE 3.7

A control volume of a nozzle section has surface pressures of 40 lbf/in^{2}absolute at section 1 and atmospheric pressure of 15 lbf/in^{2}absolute at section 2 and on the external rounded part of the nozzle, as in Fig. E3.7*a*. Compute the net pressure force if $D_1 = 3$ in and $D_2 = 1$ in.

Solution

We do not have to bother with the outer surface if we subtract 15 lbf/in^2 from all surfaces. This leaves 25 lbf/in^{2}gage at section 1 and 0 lbf/in^2 gage everywhere else, as in Fig. E3.7*b*.

[8]Can you prove this? It is a consequence of Gauss' theorem from vector analysis.



E3.7

Then the net pressure force is computed from section 1 only

$$\mathbf{F} = p_{g1}(-\mathbf{n})_1 A_1 = (25\ \text{lbf/in}^2)\frac{\pi}{4}(3\ \text{in})^2\mathbf{i} = 177\mathbf{i}\ \text{lbf} \qquad \textit{Ans.}$$

Notice that we did not change inches to feet in this case because, with pressure in pounds-force per square inch and area in square inches, the product gives force directly in pounds. More often, though, the change back to standard units is necessary and desirable. *Note:* This problem computes pressure force only. There are probably other forces involved in Fig. E3.7, e.g., nozzle-wall stresses in the cuts through sections 1 and 2 and the weight of the fluid within the control volume.

Pressure Condition at a Jet Exit

Figure E3.7 illustrates a pressure boundary condition commonly used for jet exit-flow problems. When a fluid flow leaves a confined internal duct and exits into an ambient "atmosphere," its free surface is exposed to that atmosphere. Therefore the jet itself will essentially be at atmospheric pressure also. This condition was used at section 2 in Fig. E3.7.

Only two effects could maintain a pressure difference between the atmosphere and a free exit jet. The first is surface tension, Eq. (1.31), which is usually negligible. The second effect is a *supersonic* jet, which can separate itself from an atmosphere with expansion or compression waves (Chap. 9). For the majority of applications, therefore, we shall set the pressure in an exit jet as atmospheric.

EXAMPLE 3.8

A fixed control volume of a streamtube in steady flow has a uniform inlet flow (ρ_1, A_1, V_1) and a uniform exit flow (ρ_2, A_2, V_2), as shown in Fig. 3.8. Find an expression for the net force on the control volume.

Solution

Equation (3.40) applies with one inlet and exit

$$\sum \mathbf{F} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1 = (\rho_2 A_2 V_2)\mathbf{V}_2 - (\rho_1 A_1 V_1)\mathbf{V}_1$$

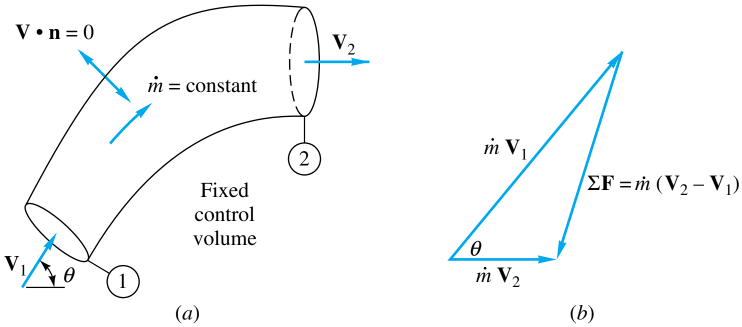


Fig. 3.8 Net force on a one-dimensional streamtube in steady flow: (*a*) streamtube in steady flow; (*b*) vector diagram for computing net force.

The volume-integral term vanishes for steady flow, but from conservation of mass in Example 3.3 we saw that

$$\dot{m}_1 = \dot{m}_2 = \dot{m} = \text{const}$$

Therefore a simple form for the desired result is

$$\sum \mathbf{F} = \dot{m}\,(\mathbf{V}_2 - \mathbf{V}_1) \qquad \textit{Ans.}$$

This is a *vector* relation and is sketched in Fig. 3.8*b*. The term Σ **F** represents the net force acting on the control volume due to all causes; it is needed to balance the change in momentum of the fluid as it turns and decelerates while passing through the control volume.

EXAMPLE 3.9

As shown in Fig. 3.9*a*, a fixed vane turns a water jet of area A through an angle θ without changing its velocity magnitude. The flow is steady, pressure is p_a everywhere, and friction on the vane is negligible. (*a*) Find the components F_x and F_y of the applied vane force. (*b*) Find expressions for the force magnitude F and the angle ϕ between F and the horizontal; plot them versus θ.

Solution

Part (a)

The control volume selected in Fig. 3.9*a* cuts through the inlet and exit of the jet and through the vane support, exposing the vane force **F**. Since there is no cut along the vane-jet interface,

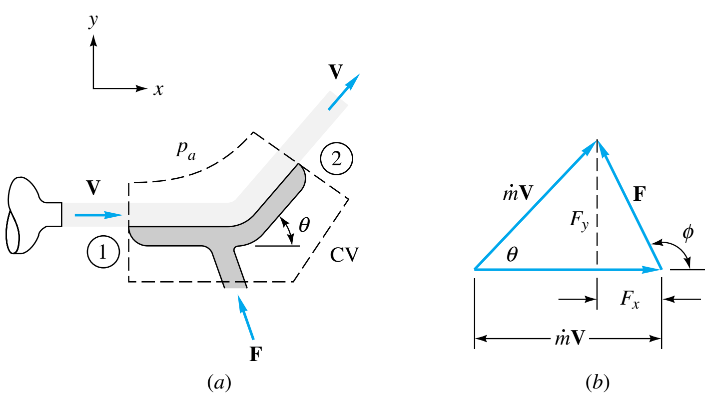


Fig. 3.9 Net applied force on a fixed jet-turning vane: (*a*) geometry of the vane turning the water jet; (*b*) vector diagram for the net force.

vane friction is internally self-canceling. The pressure force is zero in the uniform atmosphere. We neglect the weight of fluid and the vane weight within the control volume. Then Eq. (3.40) reduces to

$$\mathbf{F}_{\text{vane}} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1$$

But the magnitude $V_1 = V_2 = V$ as given, and conservation of mass for the streamtube requires $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho AV$. The vector diagram for force and momentum change becomes an isosceles triangle with legs $\dot{m}\mathbf{V}$ and base $\mathbf{F}$, as in Fig. 3.9*b*. We can readily find the force components from this diagram

$$F_x = \dot{m}V(\cos\theta - 1) \qquad F_y = \dot{m}V\sin\theta \qquad \textit{Ans. (a)}$$

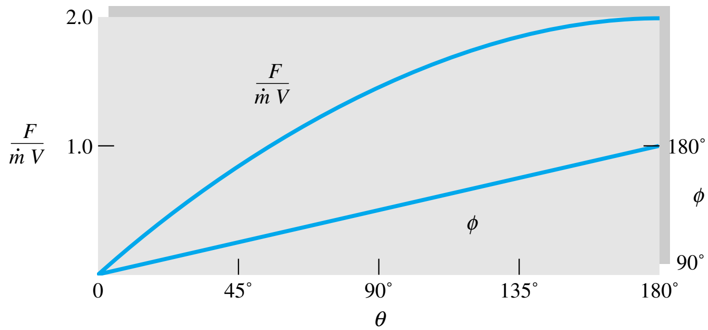
where $\dot{m}V = \rho AV^2$ for this case. This is the desired result.

Part (b) The force magnitude is obtained from part (*a*):

$$F = (F_x^2 + F_y^2)^{1/2} = \dot{m}V[\sin^2\theta + (\cos\theta - 1)^2]^{1/2} = 2\dot{m}V\sin\frac{\theta}{2} \qquad \textit{Ans. (b)}$$

From the geometry of Fig. 3.9*b* we obtain

$$\phi = 180° - \tan^{-1}\frac{F_y}{F_x} = 90° + \frac{\theta}{2} \qquad \textit{Ans. (b)}$$



E3.9

These can be plotted versus θ as shown in Fig. E3.9. Two special cases are of interest. First, the maximum force occurs at $\theta = 180°$, that is, when the jet is turned around and thrown back in the opposite direction with its momentum completely reversed. This force is $2\dot{m}V$ and acts to the *left*; that is, $\phi = 180°$. Second, at very small turning angles ($\theta < 10°$) we obtain approximately

$$F \approx \dot{m}V\theta \qquad \phi \approx 90°$$

The force is linearly proportional to the turning angle and acts nearly normal to the jet. This is the principle of a lifting vane, or airfoil, which causes a slight change in the oncoming flow direction and thereby creates a lift force normal to the basic flow.

EXAMPLE 3.10

A water jet of velocity V_j impinges normal to a flat plate which moves to the right at velocity V_c, as shown in Fig. 3.10*a*. Find the force required to keep the plate moving at constant velocity if the jet density is 1000 kg/m^3, the jet area is 3 cm^2, and V_j and V_c are 20 and 15 m/s, re-

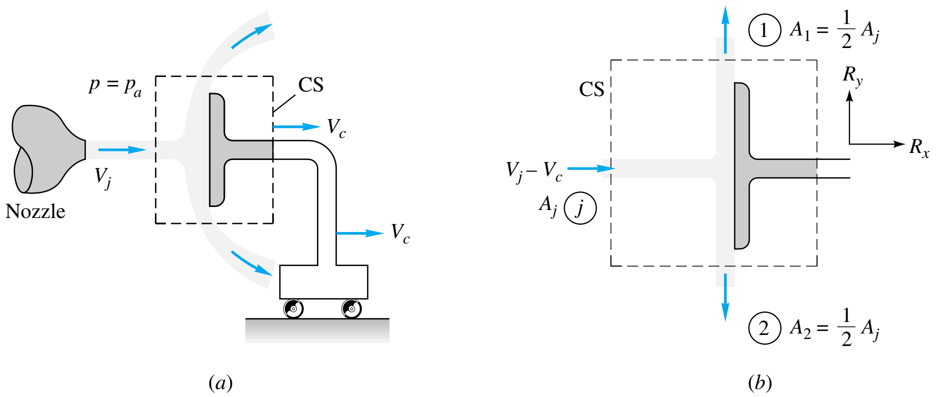


Fig. 3.10 Force on a plate moving at constant velocity: (*a*) jet striking a moving plate normally; (*b*) control volume fixed relative to the plate.

spectively. Neglect the weight of the jet and plate, and assume steady flow with respect to the moving plate with the jet splitting into an equal upward and downward half-jet.

Solution

The suggested control volume in Fig. 3.10*a* cuts through the plate support to expose the desired forces R_x and R_y. This control volume moves at speed V_c and thus is fixed relative to the plate, as in Fig. 3.10*b*. We must satisfy both mass and momentum conservation for the assumed steady-flow pattern in Fig. 3.10*b*. There are two outlets and one inlet, and Eq. (3.30) applies for mass conservation

$$\dot{m}_{\text{out}} = \dot{m}_{\text{in}}$$

or

$$\rho_1 A_1 V_1 + \rho_2 A_2 V_2 = \rho_j A_j (V_j - V_c) \quad (1)$$

We assume that the water is incompressible $\rho_1 = \rho_2 = \rho_j$, and we are given that $A_1 = A_2 = \frac{1}{2}A_j$. Therefore Eq. (1) reduces to

$$V_1 + V_2 = 2(V_j - V_c) \quad (2)$$

Strictly speaking, this is all that mass conservation tells us. However, from the symmetry of the jet deflection and the neglect of fluid weight, we conclude that the two velocities V_1 and V_2 must be equal, and hence (2) becomes

$$V_1 = V_2 = V_j - V_c \quad (3)$$

For the given numerical values, we have

$$V_1 = V_2 = 20 - 15 = 5 \text{ m/s}$$

Now we can compute R_x and R_y from the two components of momentum conservation. Equation (3.40) applies with the unsteady term zero

$$\sum F_x = R_x = \dot{m}_1 u_1 + \dot{m}_2 u_2 - \dot{m}_j u_j \quad (4)$$

where from the mass analysis, $\dot{m}_1 = \dot{m}_2 = \frac{1}{2}\dot{m}_j = \frac{1}{2}\rho_j A_j (V_j - V_c)$. Now check the flow directions at each section: $u_1 = u_2 = 0$, and $u_j = V_j - V_c = 5$ m/s. Thus Eq. (4) becomes

$$R_x = -\dot{m}_j u_j = -[\rho_j A_j (V_j - V_c)](V_j - V_c) \quad (5)$$

For the given numerical values we have

$$R_x = -(1000 \text{ kg/m}^3)(0.0003 \text{ m}^2)(5 \text{ m/s})^2 = -7.5 \text{ (kg} \cdot \text{m)/s}^2 = -7.5 \text{ N} \qquad \textit{Ans.}$$

This acts to the *left*; i.e., it requires a restraining force to keep the plate from accelerating to the right due to the continuous impact of the jet. The vertical force is

$$F_y = R_y = \dot{m}_1 \upsilon_1 + \dot{m}_2 \upsilon_2 - \dot{m}_j \upsilon_j$$

Check directions again: $\upsilon_1 = V_1$, $\upsilon_2 = -V_2$, $\upsilon_j = 0$. Thus

$$R_y = \dot{m}_1(V_1) + \dot{m}_2(-V_2) = \tfrac{1}{2}\dot{m}_j(V_1 - V_2) \qquad (6)$$

But since we found earlier that $V_1 = V_2$, this means that $R_y = 0$, as we could expect from the symmetry of the jet deflection.[9] Two other results are of interest. First, the relative velocity at section 1 was found to be 5 m/s up, from Eq. (3). If we convert this to absolute motion by adding on the control-volume speed $V_c = 15$ m/s to the right, we find that the absolute velocity $\mathbf{V}_1 = 15\mathbf{i} + 5\mathbf{j}$ m/s, or 15.8 m/s at an angle of 18.4° upward, as indicated in Fig. 3.10*a*. Thus the absolute jet speed changes after hitting the plate. Second, the computed force R_x does not change if we assume the jet deflects in all radial directions along the plate surface rather than just up and down. Since the plate is normal to the *x* axis, there would still be zero outlet *x*-momentum flux when Eq. (4) was rewritten for a radial-deflection condition.

EXAMPLE 3.11

The previous example treated a plate at normal incidence to an oncoming flow. In Fig. 3.11 the plate is parallel to the flow. The stream is not a jet but a broad river, or *free stream*, of uniform velocity $\mathbf{V} = U_0\mathbf{i}$. The pressure is assumed uniform, and so it has no net force on the plate. The plate does not block the flow as in Fig. 3.10, so that the only effect is due to boundary shear, which was neglected in the previous example. The no-slip condition at the wall brings the fluid there to a halt, and these slowly moving particles retard their neighbors above, so that at the end of the plate there is a significant retarded shear layer, or *boundary layer*, of thickness $y = \delta$. The

[9]Symmetry can be a powerful tool if used properly. Try to learn more about the uses and misuses of symmetry conditions. Here we doggedly computed the results without invoking symmetry.

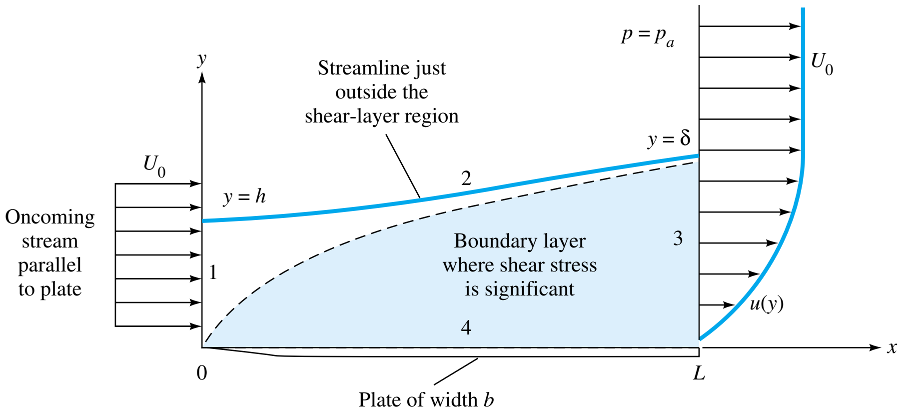


Fig. 3.11 Control-volume analysis of drag force on a flat plate due to boundary shear.

viscous stresses along the wall can sum to a finite drag force on the plate. These effects are illustrated in Fig. 3.11. The problem is to make an integral analysis and find the drag force D in terms of the flow properties ρ, U_0, and δ and the plate dimensions L and b.†

Solution

Like most practical cases, this problem requires a combined mass and momentum balance. A proper selection of control volume is essential, and we select the four-sided region from 0 to h to δ to L and back to the origin 0, as shown in Fig. 3.11. Had we chosen to cut across horizontally from left to right along the height $y = h$, we would have cut through the shear layer and exposed unknown shear stresses. Instead we follow the streamline passing through $(x, y) = (0, h)$, which is outside the shear layer and also has no mass flow across it. The four control-volume sides are thus

1. From (0, 0) to (0, h): a one-dimensional inlet, $\mathbf{V} \cdot \mathbf{n} = -U_0$
2. From (0, h) to (L, δ): a streamline, no shear, $\mathbf{V} \cdot \mathbf{n} \equiv 0$
3. From (L, δ) to (L, 0): a two-dimensional outlet, $\mathbf{V} \cdot \mathbf{n} = +u(y)$
4. From (L, 0) to (0, 0): a streamline just above the plate surface, $\mathbf{V} \cdot \mathbf{n} = 0$, shear forces summing to the drag force $-D\mathbf{i}$ acting from the plate onto the retarded fluid

The pressure is uniform, and so there is no net pressure force. Since the flow is assumed incompressible and steady, Eq. (3.37) applies with no unsteady term and fluxes only across sections 1 and 3:

$$\sum F_x = -D = \rho \int_1 u(\mathbf{V} \cdot \mathbf{n})\, dA + \rho \int_3 u(\mathbf{V} \cdot \mathbf{n})\, dA$$

$$= \rho \int_0^h U_0(-U_0) b\, dy + \rho \int_0^\delta u(+u) b\, dy$$

Evaluating the first integral and rearranging give

$$D = \rho U_0^2 b h - \rho b \int_0^\delta u^2 dy \qquad (1)$$

This could be considered the answer to the problem, but it is not useful because the height h is not known with respect to the shear-layer thickness δ. This is found by applying mass conservation, since the control volume forms a streamtube

$$\rho \int_{CS} (\mathbf{V} \cdot \mathbf{n})\, dA = 0 = \rho \int_0^h (-U_0) b\, dy + \rho \int_0^\delta u b\, dy$$

or

$$U_0 h = \int_0^\delta u\, dy \qquad (2)$$

after canceling b and ρ and evaluating the first integral. Introduce this value of h into Eq. (1) for a much cleaner result

$$D = \rho b \int_0^\delta u(U_0 - u)\, dy \Big|_{x=L} \qquad \textit{Ans. } (3)$$

This result was first derived by Theodore von Kármán in 1921.[10] It relates the friction drag on

†The general analysis of such wall-shear problems, called *boundary-layer theory*, is treated in Sec. 7.3.

[10]The autobiography of this great twentieth-century engineer and teacher [2] is recommended for its historical and scientific insight.

one side of a flat plate to the integral of the *momentum defect* $u(U_0 - u)$ across the trailing cross section of the flow past the plate. Since $U_0 - u$ vanishes as y increases, the integral has a finite value. Equation (3) is an example of *momentum-integral theory* for boundary layers, which is treated in Chap. 7. To illustrate the magnitude of this drag force, we can use a simple parabolic approximation for the outlet-velocity profile $u(y)$ which simulates low-speed, or *laminar*, shear flow

$$u \approx U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \qquad \text{for } 0 \le y \le \delta \tag{4}$$

Substituting into Eq. (3) and letting $\eta = y/\delta$ for convenience, we obtain

$$D = \rho b U_0^2 \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \tfrac{2}{15}\rho U_0^2 b\delta \tag{5}$$

This is within 1 percent of the accepted result from laminar boundary-layer theory (Chap. 7) in spite of the crudeness of the Eq. (4) approximation. This is a happy situation and has led to the wide use of Kármán's integral theory in the analysis of viscous flows. Note that D increases with the shear-layer thickness δ, which itself increases with plate length and the viscosity of the fluid (see Sec. 7.4).

Momentum-Flux Correction Factor

For flow in a duct, the axial velocity is usually nonuniform, as in Example 3.4. For this case the simple momentum-flux calculation $\int u\rho(\mathbf{V}\cdot\mathbf{n})\, dA = \dot{m}V = \rho A V^2$ is somewhat in error and should be corrected to $\beta\rho A V^2$, where β is the dimensionless momentum-flux correction factor, $\beta \ge 1$.

The factor β accounts for the variation of u^2 across the duct section. That is, we compute the exact flux and set it equal to a flux based on average velocity in the duct

$$\rho\int u^2 dA = \beta\dot{m}V_{av} = \beta\rho A V_{av}^2$$

or

$$\beta = \frac{1}{A}\int \left(\frac{u}{V_{av}}\right)^2 dA \tag{3.43a}$$

Values of β can be computed based on typical duct velocity profiles similar to those in Example 3.4. The results are as follows:

Laminar flow:
$$u = U_0\left(1 - \frac{r^2}{R^2}\right) \qquad \beta = \frac{4}{3} \tag{3.43b}$$

Turbulent flow:
$$u \approx U_0\left(1 - \frac{r}{R}\right)^m \qquad \frac{1}{9} \le m \le \frac{1}{5}$$

$$\beta = \frac{(1+m)^2(2+m)^2}{2(1+2m)(2+2m)} \tag{3.43c}$$

The turbulent correction factors have the following range of values:

Turbulent flow:

m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
β	1.037	1.027	1.020	1.016	1.013

These are so close to unity that they are normally neglected. The laminar correction may sometimes be important.

To illustrate a typical use of these correction factors, the solution to Example 3.8 for nonuniform velocities at sections 1 and 2 would be given as

$$\sum \mathbf{F} = \dot{m}(\beta_2 \mathbf{V}_2 - \beta_1 \mathbf{V}_1) \tag{3.43d}$$

Note that the basic parameters and vector character of the result are not changed at all by this correction.

Noninertial Reference Frame[11]

All previous derivations and examples in this section have assumed that the coordinate system is inertial, i.e., at rest or moving at constant velocity. In this case the rate of change of velocity equals the absolute acceleration of the system, and Newton's law applies directly in the form of Eqs. (3.2) and (3.35).

In many cases it is convenient to use a *noninertial*, or accelerating, coordinate system. An example would be coordinates fixed to a rocket during takeoff. A second example is any flow on the earth's surface, which is accelerating relative to the fixed stars because of the rotation of the earth. Atmospheric and oceanographic flows experience the so-called *Coriolis acceleration*, outlined below. It is typically less than $10^{-5}g$, where g is the acceleration of gravity, but its accumulated effect over distances of many kilometers can be dominant in geophysical flows. By contrast, the Coriolis acceleration is negligible in small-scale problems like pipe or airfoil flows.

Suppose that the fluid flow has velocity $\mathbf{V}$ relative to a noninertial xyz coordinate system, as shown in Fig. 3.12. Then $d\mathbf{V}/dt$ will represent a noninertial acceleration which must be added vectorially to a relative acceleration $\mathbf{a}_{rel}$ to give the absolute acceleration $\mathbf{a}_i$ relative to some inertial coordinate system XYZ, as in Fig. 3.12. Thus

$$\mathbf{a}_i = \frac{d\mathbf{V}}{dt} + \mathbf{a}_{rel} \tag{3.44}$$

[11]This section may be omitted without loss of continuity.

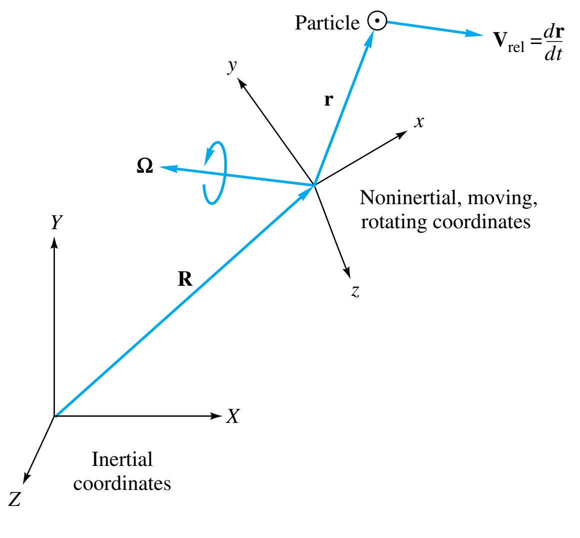


Fig. 3.12 Geometry of fixed versus accelerating coordinates.

Since Newton's law applies to the absolute acceleration,

$$\sum \mathbf{F} = m\mathbf{a}_i = m\left(\frac{d\mathbf{V}}{dt} + \mathbf{a}_{\text{rel}}\right)$$

or

$$\sum \mathbf{F} - m\mathbf{a}_{\text{rel}} = m\,\frac{d\mathbf{V}}{dt} \tag{3.45}$$

Thus Newton's law in noninertial coordinates xyz is equivalent to adding more "force" terms $-m\mathbf{a}_{\text{rel}}$ to account for noninertial effects. In the most general case, sketched in Fig. 3.12, the term $\mathbf{a}_{\text{rel}}$ contains four parts, three of which account for the angular velocity $\boldsymbol{\Omega}(t)$ of the inertial coordinates. By inspection of Fig. 3.12, the absolute displacement of a particle is

$$\mathbf{S}_i = \mathbf{r} + \mathbf{R} \tag{3.46}$$

Differentiation gives the absolute velocity

$$\mathbf{V}_i = \mathbf{V} + \frac{d\mathbf{R}}{dt} + \boldsymbol{\Omega} \times \mathbf{r} \tag{3.47}$$

A second differentiation gives the absolute acceleration:

$$\mathbf{a}_i = \frac{d\mathbf{V}}{dt} + \frac{d^2\mathbf{R}}{dt^2} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{3.48}$$

By comparison with Eq. (3.44), we see that the last four terms on the right represent the additional relative acceleration:

1. $d^2\mathbf{R}/dt^2$ is the acceleration of the noninertial origin of coordinates xyz.
2. $(d\boldsymbol{\Omega}/dt) \times \mathbf{r}$ is the angular-acceleration effect.
3. $2\boldsymbol{\Omega} \times \mathbf{V}$ is the Coriolis acceleration.
4. $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ is the centripetal acceleration, directed from the particle normal to the axis of rotation with magnitude $\Omega^2 L$, where L is the normal distance to the axis.[12]

Equation (3.45) differs from Eq. (3.2) only in the added inertial forces on the left-hand side. Thus the control-volume formulation of linear momentum in noninertial coordinates merely adds inertial terms by integrating the added relative acceleration over each differential mass in the control volume

$$\sum \mathbf{F} - \int_{\text{CV}} \mathbf{a}_{\text{rel}}\, dm = \frac{d}{dt}\left(\int_{\text{CV}} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{\text{CS}} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \tag{3.49}$$

where

$$\mathbf{a}_{\text{rel}} = \frac{d^2\mathbf{R}}{dt^2} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

This is the noninertial equivalent to the inertial form given in Eq. (3.35). To analyze such problems, one must have knowledge of the displacement $\mathbf{R}$ and angular velocity $\boldsymbol{\Omega}$ of the noninertial coordinates.

If the control volume is nondeformable, Eq. (3.49) reduces to

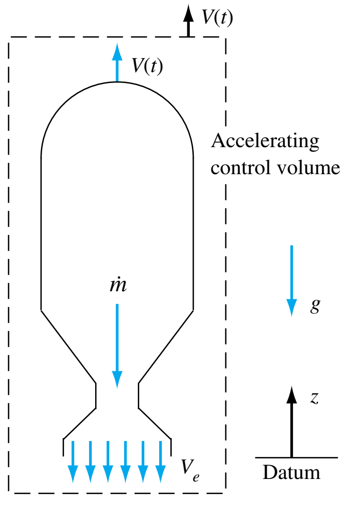
[12]A complete discussion of these noninertial coordinate terms is given, e.g., in Ref. 4, pp. 49 – 51.

$$\sum \mathbf{F} - \int_{CV} \mathbf{a}_{rel}\, dm = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})\, dA \tag{3.50}$$

In other words, the right-hand side reduces to that of Eq. (3.37).

EXAMPLE 3.12

A classic example of an accelerating control volume is a rocket moving straight up, as in Fig. E3.12. Let the initial mass be M_0, and assume a steady exhaust mass flow $\dot{m}$ and exhaust velocity V_e relative to the rocket, as shown. If the flow pattern within the rocket motor is steady and air drag is neglected, derive the differential equation of vertical rocket motion $V(t)$ and integrate using the initial condition $V = 0$ at $t = 0$.



E3.12

Solution

The appropriate control volume in Fig. E3.12 encloses the rocket, cuts through the exit jet, and accelerates upward at rocket speed $V(t)$. The z-momentum equation (3.49) becomes

$$\sum F_z - \int a_{rel}\, dm = \frac{d}{dt}\left(\int_{CV} w\, d\dot{m}\right) + (\dot{m}w)_e$$

or $$-mg - m\frac{dV}{dt} = 0 + \dot{m}V_e \qquad \text{with} \quad m = m(t) = M_0 - \dot{m}t$$

The term $a_{rel} = dV/dt$ of the rocket. The control volume integral vanishes because of the steady rocket-flow conditions. Separate the variables and integrate, assuming $V = 0$ at $t = 0$:

$$\int_0^V dV = \dot{m}V_e \int_0^t \frac{dt}{M_0 - \dot{m}t} - g\int_0^t dt \quad \text{or} \quad V(t) = -V_e \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - gt \qquad \textit{Ans.}$$

This is a classic approximate formula in rocket dynamics. The first term is positive and, if the fuel mass burned is a large fraction of initial mass, the final rocket velocity can exceed V_e.

3.5 The Angular-Momentum Theorem[13]

A control-volume analysis can be applied to the angular-momentum relation, Eq. (3.3), by letting our dummy variable **B** be the angular-momentum vector **H**. However, since the system considered here is typically a group of nonrigid fluid particles of variable velocity, the concept of mass moment of inertia is of no help and we have to calculate the instantaneous angular momentum by integration over the elemental masses dm. If O is the point about which moments are desired, the angular momentum about O is given by

$$\mathbf{H}_O = \int_{syst} (\mathbf{r} \times \mathbf{V})\, dm \tag{3.51}$$

where **r** is the position vector from 0 to the elemental mass dm and **V** is the velocity of that element. The amount of angular momentum per unit mass is thus seen to be

$$\boldsymbol{\beta} = \frac{d\mathbf{H}_O}{dm} = \mathbf{r} \times \mathbf{V}$$

[13]This section may be omitted without loss of continuity.

The Reynolds transport theorem (3.16) then tells us that

$$\left.\frac{d\mathbf{H}_O}{dt}\right|_{\text{syst}} = \frac{d}{dt}\left[\int_{\text{CV}} (\mathbf{r} \times \mathbf{V})\rho \, d\mathcal{V}\right] + \int_{\text{CS}} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA \qquad (3.52)$$

for the most general case of a deformable control volume. But from the angular-momentum theorem (3.3), this must equal the sum of all the moments about point O applied to the control volume

$$\frac{d\mathbf{H}_O}{dt} = \sum \mathbf{M}_O = \sum (\mathbf{r} \times \mathbf{F})_O$$

Note that the total moment equals the summation of moments of all applied forces about point O. Recall, however, that this law, like Newton's law (3.2), assumes that the particle velocity **V** is relative to an *inertial* coordinate system. If not, the moments about point O of the relative acceleration terms $\mathbf{a}_{\text{rel}}$ in Eq. (3.49) must also be included

$$\sum \mathbf{M}_O = \sum (\mathbf{r} \times \mathbf{F})_O - \int_{\text{CV}} (\mathbf{r} \times \mathbf{a}_{\text{rel}}) \, dm \qquad (3.53)$$

where the four terms constituting $\mathbf{a}_{\text{rel}}$ are given in Eq. (3.49). Thus the most general case of the angular-momentum theorem is for a deformable control volume associated with a noninertial coordinate system. We combine Eqs. (3.52) and (3.53) to obtain

$$\sum (\mathbf{r} \times \mathbf{F})_0 - \int_{\text{CV}} (\mathbf{r} \times \mathbf{a}_{\text{rel}}) \, dm = \frac{d}{dt}\left[\int_{\text{CV}} (\mathbf{r} \times \mathbf{V})\rho \, d\mathcal{V}\right] + \int_{\text{CS}} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V}_r \cdot \mathbf{n}) \, dA \qquad (3.54)$$

For a nondeformable inertial control volume, this reduces to

$$\sum \mathbf{M}_0 = \frac{\partial}{\partial t}\left[\int_{\text{CV}} (\mathbf{r} \times \mathbf{V})\rho \, d\mathcal{V}\right] + \int_{\text{CS}} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \qquad (3.55)$$

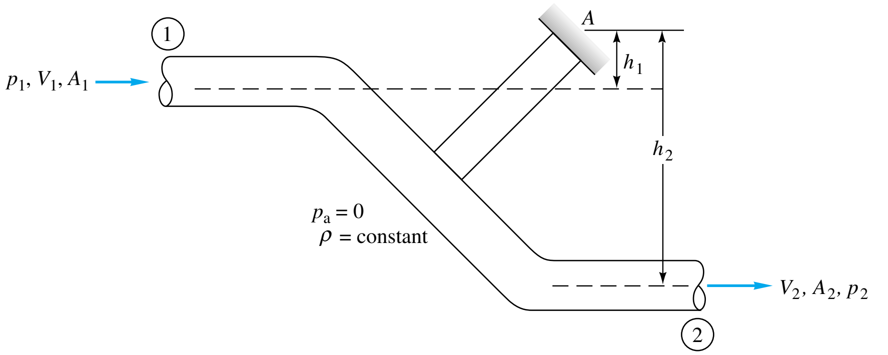
Further, if there are only one-dimensional inlets and exits, the angular-momentum flux terms evaluated on the control surface become

$$\int_{\text{CS}} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) \, dA = \sum (\mathbf{r} \times \mathbf{V})_{\text{out}} \, \dot{m}_{\text{out}} - \sum (\mathbf{r} \times \mathbf{V})_{\text{in}} \, \dot{m}_{\text{in}} \qquad (3.56)$$

Although at this stage the angular-momentum theorem can be considered to be a supplementary topic, it has direct application to many important fluid-flow problems involving torques or moments. A particularly important case is the analysis of rotating fluid-flow devices, usually called *turbomachines* (Chap. 11).

EXAMPLE 3.13

As shown in Fig. E3.13*a*, a pipe bend is supported at point A and connected to a flow system by flexible couplings at sections 1 and 2. The fluid is incompressible, and ambient pressure p_a is zero. (*a*) Find an expression for the torque T which must be resisted by the support at A, in terms of the flow properties at sections 1 and 2 and the distances h_1 and h_2. (*b*) Compute this torque if $D_1 = D_2 = 3$ in, $p_1 = 100$ lbf/in^2 gage, $p_2 = 80$ lbf/in^2 gage, $V_1 = 40$ ft/s, $h_1 = 2$ in, $h_2 = 10$ in, and $\rho = 1.94$ slugs/ft^3.



E3.13a

Solution

Part (a) The control volume chosen in Fig. E3.13*b* cuts through sections 1 and 2 and through the support at A, where the torque T_A is desired. The flexible-couplings description specifies that there is no torque at either section 1 or 2, and so the cuts there expose no moments. For the angular-momentum terms $\mathbf{r} \times \mathbf{V}$, $\mathbf{r}$ should be taken from point A to sections 1 and 2. Note that the gage pressure forces p_1A_1 and p_2A_2 both have moments about A. Equation (3.55) with one-dimensional flux terms becomes

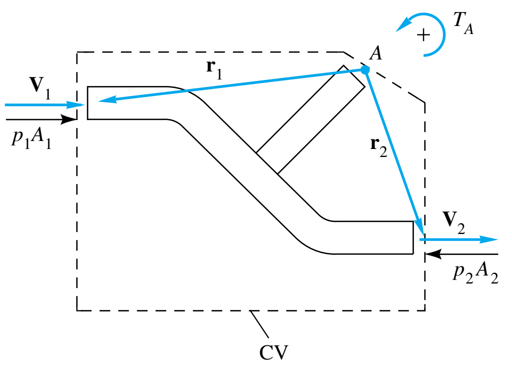
$$\sum \mathbf{M}_A = \mathbf{T}_A + \mathbf{r}_1 \times (-p_1A_1\mathbf{n}_1) + \mathbf{r}_2 \times (-p_2A_2\mathbf{n}_2)$$
$$= (\mathbf{r}_2 \times \mathbf{V}_2)(+\dot{m}_{out}) + (\mathbf{r}_1 \times \mathbf{V}_1)(-\dot{m}_{in}) \qquad (1)$$

Figure E3.13*c* shows that all the cross products are associated either with $r_1 \sin \theta_1 = h_1$ or $r_2 \sin \theta_2 = h_2$, the perpendicular distances from point A to the pipe axes at 1 and 2. Remember that $\dot{m}_{in} = \dot{m}_{out}$ from the steady-flow continuity relation. In terms of counterclockwise moments, Eq. (1) then becomes

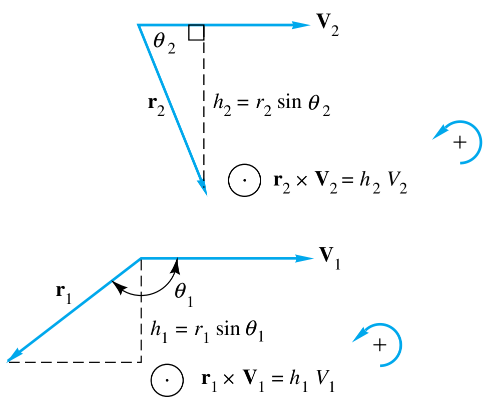
$$T_A + p_1A_1h_1 - p_2A_2h_2 = \dot{m}(h_2V_2 - h_1V_1) \qquad (2)$$

Rewriting this, we find the desired torque to be

$$T_A = h_2(p_2A_2 + \dot{m}V_2) - h_1(p_1A_1 + \dot{m}V_1) \qquad \textit{Ans. (a)}\ (3)$$



E3.13b



E3.13c

counterclockwise. The quantities p_1 and p_2 are gage pressures. Note that this result is independent of the shape of the pipe bend and varies only with the properties at sections 1 and 2 and the distances h_1 and h_2.[†]

Part (b) The inlet and exit areas are the same:

$$A_1 = A_2 = \frac{\pi}{4}(3)^2 = 7.07 \text{ in}^2 = 0.0491 \text{ ft}^2$$

Since the density is constant, we conclude from continuity that $V_2 = V_1 = 40$ ft/s. The mass flow is

$$\dot{m} = \rho A_1 V_1 = 1.94(0.0491)(40) = 3.81 \text{ slug/s}$$

Equation (3) can be evaluated as

$$T_A = (\tfrac{10}{12}\text{ ft})[80(7.07)\text{ lbf} + 3.81(40)\text{ lbf}] - (\tfrac{2}{12}\text{ ft})[100(7.07)\text{ lbf} + 3.81(40)\text{ lbf}]$$

$$= 598 - 143 = 455 \text{ ft} \cdot \text{lbf counterclockwise} \qquad \textit{Ans. (b)}$$

We got a little daring there and multiplied p in lbf/in^2 gage times A in in^2 to get lbf without changing units to lbf/ft^2 and ft^2.

EXAMPLE 3.14

Figure 3.13 shows a schematic of a centrifugal pump. The fluid enters axially and passes through the pump blades, which rotate at angular velocity ω; the velocity of the fluid is changed from V_1 to V_2 and its pressure from p_1 to p_2. (*a*) Find an expression for the torque T_O which must be applied to these blades to maintain this flow. (*b*) The power supplied to the pump would be $P = \omega T_O$. To illustrate numerically, suppose $r_1 = 0.2$ m, $r_2 = 0.5$ m, and $b = 0.15$ m. Let the pump rotate at 600 r/min and deliver water at 2.5 m^3/s with a density of 1000 kg/m^3. Compute the idealized torque and power supplied.

Solution

Part (a) The control volume is chosen to be the angular region between sections 1 and 2 where the flow passes through the pump blades (see Fig. 3.13). The flow is steady and assumed incompressible. The contribution of pressure to the torque about axis O is zero since the pressure forces at 1 and 2 act radially through O. Equation (3.55) becomes

$$\sum \mathbf{M}_O = \mathbf{T}_O = (\mathbf{r}_2 \times \mathbf{V}_2)\dot{m}_{out} - (\mathbf{r}_1 \times \mathbf{V}_1)\dot{m}_{in} \qquad (1)$$

where steady-flow continuity tells us that

$$\dot{m}_{in} = \rho V_{n1} 2\pi r_1 b = \dot{m}_{out} = \rho V_{n2} \pi r_2 b = \rho Q$$

The cross product $\mathbf{r} \times \mathbf{V}$ is found to be clockwise about O at both sections:

$$\mathbf{r}_2 \times \mathbf{V}_2 = r_2 V_{t2} \sin 90° \,\mathbf{k} = r_2 V_{t2} \boldsymbol{k} \qquad \text{clockwise}$$

$$\mathbf{r}_1 \times \mathbf{V}_1 = r_1 V_{t1} \boldsymbol{k} \qquad \text{clockwise}$$

Equation (1) thus becomes the desired formula for torque

$$\boldsymbol{T}_O = \rho Q(r_2 V_{t2} - r_1 V_{t1})\mathbf{k} \qquad \text{clockwise} \qquad \textit{Ans. (a)} \qquad (2a)$$

[†]Indirectly, the pipe-bend shape probably affects the pressure change from p_1 to p_2.

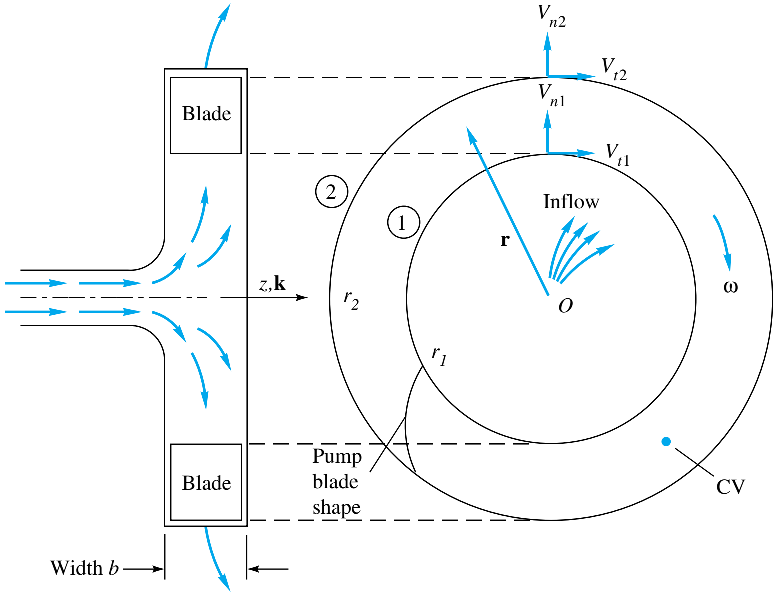


Fig. 3.13 Schematic of a simplified centrifugal pump.

This relation is called *Euler's turbine formula*. In an idealized pump, the inlet and outlet tangential velocities would match the blade rotational speeds $V_{t1} = \omega r_1$ and $V_{t2} = \omega r_2$. Then the formula for torque supplied becomes

$$T_O = \rho Q \omega (r_2^2 - r_1^2) \qquad \text{clockwise} \tag{2b}$$

Part (b) Convert ω to $600(2\pi/60) = 62.8$ rad/s. The normal velocities are not needed here but follow from the flow rate

$$V_{n1} = \frac{Q}{2\pi r_1 b} = \frac{2.5\ m^3/\text{s}}{2\pi(0.2\text{ m})(0.15\text{ m})} = 13.3\text{ m/s}$$

$$V_{n2} = \frac{Q}{2\pi r_2 b} = \frac{2.5}{2\pi(0.5)(0.15)} = 5.3\text{ m/s}$$

For the idealized inlet and outlet, tangential velocity equals tip speed

$$V_{t1} = \omega r_1 = (62.8\text{ rad/s})(0.2\text{ m}) = 12.6\text{ m/s}$$

$$V_{t2} = \omega r_2 = 62.8(0.5) = 31.4\text{ m/s}$$

Equation (2*a*) predicts the required torque to be

$$T_O = (1000\text{ kg/m}^3)(2.5\text{ m}^3/\text{s})[(0.5\text{ m})(31.4\text{ m/s}) - (0.2\text{ m})(12.6\text{ m/s})]$$

$$= 33{,}000\ (\text{kg}\cdot\text{m}^2)/\text{s}^2 = 33{,}000\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

The power required is

$$P = \omega T_O = (62.8\text{ rad/s})(33{,}000\text{ N}\cdot\text{m}) = 2{,}070{,}000\ (\text{N}\cdot\text{m})/\text{s}$$

$$= 2.07\text{ MW (2780 hp)} \qquad \textit{Ans.}$$

In actual practice the tangential velocities are considerably less than the impeller-tip speeds, and the design power requirements for this pump may be only 1 MW or less.

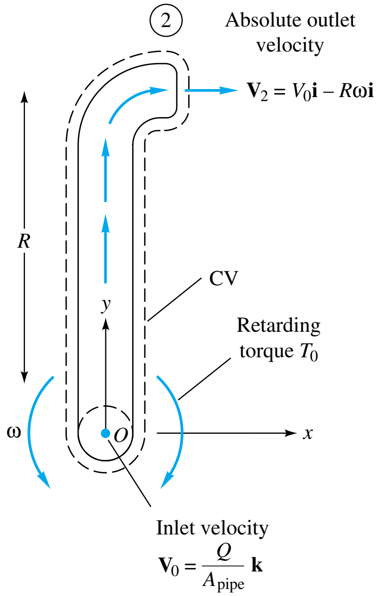


Fig. 3.14 View from above of a single arm of a rotating lawn sprinkler.

EXAMPLE 3.15

Figure 3.14 shows a lawn-sprinkler arm viewed from above. The arm rotates about O at constant angular velocity ω. The volume flux entering the arm at O is Q, and the fluid is incompressible. There is a retarding torque at O, due to bearing friction, of amount $-T_O\mathbf{k}$. Find an expression for the rotation ω in terms of the arm and flow properties.

Solution

The entering velocity is $V_0\mathbf{k}$, where $V_0 = Q/A_{\text{pipe}}$. Equation (3.55) applies to the control volume sketched in Fig. 3.14 only if $\mathbf{V}$ is the absolute velocity relative to an inertial frame. Thus the exit velocity at section 2 is

$$\mathbf{V}_2 = V_0\mathbf{i} - R\omega\mathbf{i}$$

Equation (3.55) then predicts that, for steady flow,

$$\sum \mathbf{M}_O = -T_O\mathbf{k} = (\mathbf{r}_2 \times \mathbf{V}_2)\dot{m}_{\text{out}} - (\mathbf{r}_1 \times \mathbf{V}_1)\dot{m}_{\text{in}} \qquad (1)$$

where, from continuity, $\dot{m}_{\text{out}} = \dot{m}_{\text{in}} = \rho Q$. The cross products with reference to point O are

$$\mathbf{r}_2 \times \mathbf{V}_2 = R\mathbf{j} \times (V_0 - R\omega)\mathbf{i} = (R^2\omega - RV_0)\mathbf{k}$$

$$\mathbf{r}_1 \times \mathbf{V}_1 = 0\mathbf{j} \times V_0\mathbf{k} = 0$$

Equation (1) thus becomes

$$-T_O\mathbf{k} = \rho Q(R^2\omega - RV_0)\mathbf{k}$$

$$\omega = \frac{V_O}{R} - \frac{T_O}{\rho Q R^2} \qquad \textit{Ans.}$$

The result may surprise you: Even if the retarding torque T_O is negligible, the arm rotational speed is limited to the value V_0/R imposed by the outlet speed and the arm length.

3.6 The Energy Equation[14]

As our fourth and final basic law, we apply the Reynolds transport theorem (3.12) to the first law of thermodynamics, Eq. (3.5). The dummy variable B becomes energy E, and the energy per unit mass is $\beta = dE/dm = e$. Equation (3.5) can then be written for a fixed control volume as follows:[15]

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} = \frac{d}{dt}\left(\int_{CV} e\rho \, d\mathcal{V}\right) + \int_{CS} e\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \qquad (3.57)$$

Recall that positive Q denotes heat added to the system and positive W denotes work done by the system.

The system energy per unit mass e may be of several types:

$$e = e_{\text{internal}} + e_{\text{kinetic}} + e_{\text{potential}} + e_{\text{other}}$$

[14]This section should be read for information and enrichment even if you lack formal background in thermodynamics.

[15]The energy equation for a deformable control volume is rather complicated and is not discussed here. See Refs. 4 and 5 for further details.

where e_{other} could encompass chemical reactions, nuclear reactions, and electrostatic or magnetic field effects. We neglect e_{other} here and consider only the first three terms as discussed in Eq. (1.9), with z defined as "up":

$$e = \hat{u} + \tfrac{1}{2}V^2 + gz \tag{3.58}$$

The heat and work terms could be examined in detail. If this were a heat-transfer book, *dQ/dT* would be broken down into conduction, convection, and radiation effects and whole chapters written on each (see, e.g., Ref. 3). Here we leave the term untouched and consider it only occasionally.

Using for convenience the overdot to denote the time derivative, we divide the work term into three parts:

$$\dot{W} = \dot{W}_{\text{shaft}} + \dot{W}_{\text{press}} + \dot{W}_{\text{viscous stresses}} = \dot{W}_s + \dot{W}_p + \dot{W}_v$$

The work of gravitational forces has already been included as potential energy in Eq. (3.58). Other types of work, e.g., those due to electromagnetic forces, are excluded here.

The shaft work isolates that portion of the work which is deliberately done by a machine (pump impeller, fan blade, piston, etc.) protruding through the control surface into the control volume. No further specification other than $\dot{W}_s$ is desired at this point, but calculations of the work done by turbomachines will be performed in Chap. 11.

The rate of work $\dot{W}_p$ done on pressure forces occurs at the surface only; all work on internal portions of the material in the control volume is by equal and opposite forces and is self-canceling. The pressure work equals the pressure force on a small surface element dA times the normal velocity component into the control volume

$$d\dot{W}_p = -(p\, dA)V_{n,\text{in}} = -p(-\mathbf{V} \cdot \mathbf{n})\, dA$$

The total pressure work is the integral over the control surface

$$\dot{W}_p = \int_{\text{CS}} p(\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.59}$$

A cautionary remark: If part of the control surface is the surface of a machine part, we prefer to delegate that portion of the pressure to the *shaft work* term $\dot{W}_s$, not to $\dot{W}_p$, which is primarily meant to isolate the fluid-flow pressure-work terms.

Finally, the shear work due to viscous stresses occurs at the control surface, the internal work terms again being self-canceling, and consists of the product of each viscous stress (one normal and two tangential) and the respective velocity component

$$d\dot{W}_v = -\boldsymbol{\tau} \cdot \mathbf{V}\, dA$$

or

$$\dot{W}_v = -\int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\, dA \tag{3.60}$$

where $\boldsymbol{\tau}$ is the stress vector on the elemental surface dA. This term may vanish or be negligible according to the particular type of surface at that part of the control volume:

Solid surface. For all parts of the control surface which are solid confining walls, $\mathbf{V} = 0$ from the viscous no-slip condition; hence $\dot{W}_v$ = zero identically.

Surface of a machine. Here the viscous work is contributed by the machine, and so we absorb this work in the term $\dot{W}_s$.

An inlet or outlet. At an inlet or outlet, the flow is approximately normal to the element dA; hence the only viscous-work term comes from the normal stress $\tau_{nn} V_n \, dA$. Since viscous normal stresses are extremely small in all but rare cases, e.g., the interior of a shock wave, it is customary to neglect viscous work at inlets and outlets of the control volume.

Streamline surface. If the control surface is a streamline such as the upper curve in the boundary-layer analysis of Fig. 3.11, the viscous-work term must be evaluated and retained if shear stresses are significant along this line. In the particular case of Fig. 3.11, the streamline is outside the boundary layer, and viscous work is negligible.

The net result of the above discussion is that the rate-of-work term in Eq. (3.57) consists essentially of

$$\dot{W} = \dot{W}_s + \int_{CS} p(\mathbf{V} \cdot \mathbf{n}) \, dA - \int_{CS} (\boldsymbol{\tau} \cdot \mathbf{V})_{SS} \, dA \tag{3.61}$$

where the subscript SS stands for stream surface. When we introduce (3.61) and (3.58) into (3.57), we find that the pressure-work term can be combined with the energy-flux term since both involve surface integrals of $\mathbf{V} \cdot \mathbf{n}$. The control-volume energy equation thus becomes

$$\dot{Q} - \dot{W}_s - (\dot{W}_\upsilon)_{SS} = \frac{\partial}{\partial t}\left(\int_{CV} e\rho \, d\mathcal{V}\right) + \int_{CS} \left(e + \frac{p}{\rho}\right)\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \tag{3.62}$$

Using e from (3.58), we see that the enthalpy $\hat{h} = \hat{u} + p/\rho$ occurs in the control-surface integral. The final general form for the energy equation for a fixed control volume becomes

$$\dot{Q} - \dot{W}_s - \dot{W}_\upsilon = \frac{\partial}{\partial t}\left[\int_{CV} \left(\hat{u} + \tfrac{1}{2}V^2 + gz\right)\rho \, d\mathcal{V}\right] + \int_{CS} \left(\hat{h} + \tfrac{1}{2}V^2 + gz\right)\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \tag{3.63}$$

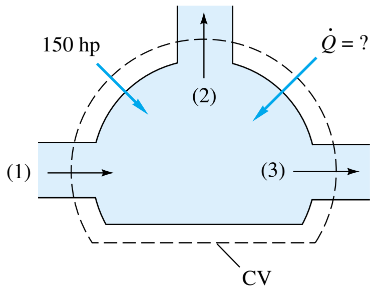
As mentioned above, the shear-work term $\dot{W}_\upsilon$ is rarely important.

One-Dimensional Energy-Flux Terms

If the control volume has a series of one-dimensional inlets and outlets, as in Fig. 3.6, the surface integral in (3.63) reduces to a summation of outlet fluxes minus inlet fluxes

$$\int_{CS} (\hat{h} + \tfrac{1}{2}V^2 + gz)\rho(\mathbf{V} \cdot \mathbf{n}) \, dA = \sum (\hat{h} + \tfrac{1}{2}V^2 + gz)_{\text{out}} \dot{m}_{\text{out}} - \sum (\hat{h} + \tfrac{1}{2}V^2 + gz)_{\text{in}} \dot{m}_{\text{in}} \tag{3.64}$$

where the values of $\hat{h}$, $\frac{1}{2}V^2$, and gz are taken to be averages over each cross section.



E3.16

EXAMPLE 3.16

A steady-flow machine (Fig. E3.16) takes in air at section 1 and discharges it at sections 2 and 3. The properties at each section are as follows:

Section	A, ft^2	Q, ft^3/s	T, °F	p, lbf/in^2 abs	z, ft
1	0.4	100	70	20	1.0
2	1.0	40	100	30	4.0
3	0.25	50	200	?	1.5

Work is provided to the machine at the rate of 150 hp. Find the pressure p_3 in lbf/in^2 absolute and the heat transfer $\dot{Q}$ in Btu/s. Assume that air is a perfect gas with $R = 1715$ and $c_p = 6003$ ft · lbf/(slug · °R).

Solution

The control volume chosen cuts across the three desired sections and otherwise follows the solid walls of the machine. Therefore the shear-work term W_v is negligible. We have enough information to compute $V_i = Q_i/A_i$ immediately

$$V_1 = \frac{100}{0.4} = 250 \text{ ft/s} \qquad V_2 = \frac{40}{1.0} = 40 \text{ ft/s} \qquad V_3 = \frac{50}{0.25} = 200 \text{ ft/s}$$

and the densities $\rho_i = p_i/(RT_i)$

$$\rho_1 = \frac{20(144)}{1715(70 + 460)} = 0.00317 \text{ slug/ft}^3$$

$$\rho_2 = \frac{30(144)}{1715(560)} = 0.00450 \text{ slug/ft}^3$$

but ρ_3 is determined from the steady-flow continuity relation:

$$\dot{m}_1 = \dot{m}_2 + \dot{m}_3$$

$$\rho_1 Q_1 = \rho_2 Q_2 + \rho_3 Q_3 \qquad (1)$$

$$0.00317(100) = 0.00450(40) + \rho_3(50)$$

or

$$50\rho_3 = 0.317 - 0.180 = 0.137 \text{ slug/s}$$

$$\rho_3 = \frac{0.137}{50} = 0.00274 \text{ slug/ft}^3 = \frac{144 p_3}{1715(660)}$$

$$p_3 = 21.5 \text{ lbf/in}^2 \text{ absolute} \qquad \textit{Ans.}$$

Note that the volume flux $Q_1 \neq Q_2 + Q_3$ because of the density changes.

For steady flow, the volume integral in (3.63) vanishes, and we have agreed to neglect viscous work. With one inlet and two outlets, we obtain

$$\dot{Q} - \dot{W}_s = -\dot{m}_1(\hat{h}_1 + \tfrac{1}{2}V_1^2 + gz_1) + \dot{m}_2(\hat{h}_2 + \tfrac{1}{2}V_2^2 + gz_2) + \dot{m}_3(\hat{h}_3 + \tfrac{1}{2}V_3^2 + gz_3) \qquad (2)$$

where $\dot{W}_s$ is given in hp and can be quickly converted to consistent BG units:

$$\dot{W}_s = -150 \text{ hp } [550 \text{ ft} \cdot \text{lbf/(s} \cdot \text{hp)}]$$

$$= -82{,}500 \text{ ft} \cdot \text{lbf/s} \qquad \text{negative work on system}$$

For a perfect gas with constant c_p, $\hat{h} = c_p T$ plus an arbitrary constant. It is instructive to separate the flux terms in Eq. (2) above to examine their magnitudes:

Enthalpy flux:

$$c_p(-\dot{m}_1 T_1 + \dot{m}_2 T_2 + \dot{m}_3 T_3) = [6003 \text{ ft} \cdot \text{lbf/(slug} \cdot {}^\circ\text{R)}][(-0.317 \text{ slug/s})(530\ {}^\circ\text{R})$$
$$+ 0.180(560) + 0.137(660)]$$
$$= -1{,}009{,}000 + 605{,}000 + 543{,}000$$
$$= +139{,}000 \text{ ft} \cdot \text{lbf/s}$$

Kinetic-energy flux:

$$-\dot{m}_1(\tfrac{1}{2}V_1^2) + \dot{m}_2(\tfrac{1}{2}V_2^2) + \dot{m}_3(\tfrac{1}{2}V_3^2) = \tfrac{1}{2}[-0.317(250)^2 + 0.180(40)^2 + 0.137(200)^2]$$
$$= -9900 + 150 + 2750 = -7000 \text{ ft} \cdot \text{lbf/s}$$

Potential-energy flux:

$$g(-\dot{m}_1 z_1 + \dot{m}_2 z_2 + \dot{m}_3 z_3) = 32.2[-0.317(1.0) + 0.180(4.0) + 0.137(1.5)]$$
$$= -10 + 23 + 7 = +20 \text{ ft} \cdot \text{lbf/s}$$

These are typical effects: The potential-energy flux is negligible in gas flows, the kinetic-energy flux is small in low-speed flows, and the enthalpy flux is dominant. It is only when we neglect heat-transfer effects that the kinetic and potential energies become important. Anyway, we can now solve for the heat flux

$$\dot{Q} = -82{,}500 + 139{,}000 - 7000 + 20 = 49{,}520 \text{ ft} \cdot \text{lbf/s} \tag{3}$$

Converting, we get

$$\dot{Q} = \frac{49{,}520}{778.2 \text{ ft} \cdot \text{lbf/Btu}} = +63.6 \text{ Btu/s} \qquad \textit{Ans.}$$

The Steady-Flow Energy Equation

For steady flow with one inlet and one outlet, both assumed one-dimensional, Eq. (3.63) reduces to a celebrated relation used in many engineering analyses. Let section 1 be the inlet and section 2 the outlet. Then

$$\dot{Q} - \dot{W}_s - \dot{W}_\upsilon = -\dot{m}_1(\hat{h}_1 + \tfrac{1}{2}V_1^2 + gz_1) + \dot{m}_2(\hat{h}_2 + \tfrac{1}{2}V_2^2 + gz_2) \tag{3.65}$$

But, from continuity, $\dot{m}_1 = \dot{m}_2 = \dot{m}$, and we can rearrange (3.65) as follows:

$$\hat{h}_1 + \tfrac{1}{2}V_1^2 + gz_1 = (\hat{h}_2 + \tfrac{1}{2}V_2^2 + gz_2) - q + w_s + w_\upsilon \tag{3.66}$$

where $q = \dot{Q}/\dot{m} = dQ/dm$, the heat transferred to the fluid per unit mass. Similarly, $w_s = \dot{W}_s/\dot{m} = dW_s/dm$ and $w_\upsilon = \dot{W}_\upsilon/\dot{m} = dW_\upsilon/dm$. Equation (3.66) is a general form of the *steady-flow energy equation*, which states that the upstream *stagnation enthalpy* $H_1 = (\hat{h} + \tfrac{1}{2}V^2 + gz)_1$ differs from the downstream value H_2 only if there is heat transfer, shaft work, or viscous work as the fluid passes between sections 1 and 2. Recall that q is positive if heat is added to the control volume and that w_s and w_υ are positive if work is done by the fluid on the surroundings.

Each term in Eq. (3.66) has the dimensions of energy per unit mass, or velocity squared, which is a form commonly used by mechanical engineers. If we divide through by g, each term becomes a length, or head, which is a form preferred by civil engineers. The traditional symbol for head is h, which we do not wish to confuse with enthalpy. Therefore we use internal energy in rewriting the head form of the energy relation:

$$\frac{p_1}{\gamma} + \frac{\hat{u}_1}{g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\hat{u}_2}{g} + \frac{V_1^2}{2g} + z_2 - h_q + h_s + h_v \tag{3.67}$$

where $h_q = q/g$, $h_s = w_s/g$, and $h_v = w_v/g$ are the head forms of the heat added, shaft work done, and viscous work done, respectively. The term p/γ is called *pressure head* and the term $V^2/2g$ is denoted as *velocity head.*

Friction Losses in Low-Speed Flow

A very common application of the steady-flow energy equation is for low-speed flow with no shaft work and negligible viscous work, such as liquid flow in pipes. For this case Eq. (3.67) may be written in the form

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \left(\frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2\right) + \frac{\hat{u}_2 - \hat{u}_1 - q}{g} \tag{3.68}$$

The term in parentheses is called the useful head or *available head* or *total head* of the flow, denoted as h_0. The last term on the right is the difference between the available head upstream and downstream and is normally *positive*, representing the loss in head due to friction, denoted as h_f. Thus, in low-speed (nearly incompressible) flow with one inlet and one exit, we may write

$$\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right)_{\text{in}} = \left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right)_{\text{out}} + h_{\text{friction}} - h_{\text{pump}} + h_{\text{turbine}} \tag{3.69}$$

Most of our internal-flow problems will be solved with the aid of Eq. (3.69). The h terms are all positive; that is, friction loss is always positive in real (viscous) flows, a pump adds energy (increases the left-hand side), and a turbine extracts energy from the flow. If h_p and/or h_t are included, the pump and/or turbine must lie *between* points 1 and 2. In Chaps. 5 and 6 we shall develop methods of correlating h_f losses with flow parameters in pipes, valves, fittings, and other internal-flow devices.

EXAMPLE 3.17

Gasoline at 20°C is pumped through a smooth 12-cm-diameter pipe 10 km long, at a flow rate of 75 m^3/h (330 gal/min). The inlet is fed by a pump at an absolute pressure of 24 atm. The exit is at standard atmospheric pressure and is 150 m higher. Estimate the frictional head loss h_f, and compare it to the velocity head of the flow $V^2/(2g)$. (These numbers are quite realistic for liquid flow through long pipelines.)

Solution

For gasoline at 20°C, from Table A.3, $\rho = 680\ \text{kg/m}^3$, or $\gamma = (680)(9.81) = 6670\ \text{N/m}^3$. There is no shaft work; hence Eq. (3.69) applies and can be evaluated:

$$\frac{p_{\text{in}}}{\gamma} + \frac{V_{\text{in}}^2}{2g} + z_{\text{in}} = \frac{p_{\text{out}}}{\gamma} + \frac{V_{\text{out}}^2}{2g} + z_{\text{out}} + h_f \qquad (1)$$

The pipe is of uniform cross section, and thus the average velocity everywhere is

$$V_{\text{in}} = V_{\text{out}} = \frac{Q}{A} = \frac{(75/3600)\ \text{m}^3/\text{s}}{(\pi/4)(0.12\ \text{m})^2} \approx 1.84\ \text{m/s}$$

Being equal at inlet and exit, this term will cancel out of Eq. (1) above, but we are asked to compute the velocity head of the flow for comparison purposes:

$$\frac{V^2}{2g} = \frac{(1.84\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} \approx 0.173\ \text{m}$$

Now we are in a position to evaluate all terms in Eq. (1) except the friction head loss:

$$\frac{(24)(101{,}350\ \text{N/m}^2)}{6670\ \text{N/m}^3} + 0.173\ \text{m} + 0\ \text{m} = \frac{101{,}350\ \text{N/m}^2}{6670\ \text{N/m}^3} + 0.173\ \text{m} + 150\ \text{m} + h_f$$

or

$$h_f = 364.7 - 15.2 - 150 \approx 199\ \text{m} \qquad \textit{Ans.}$$

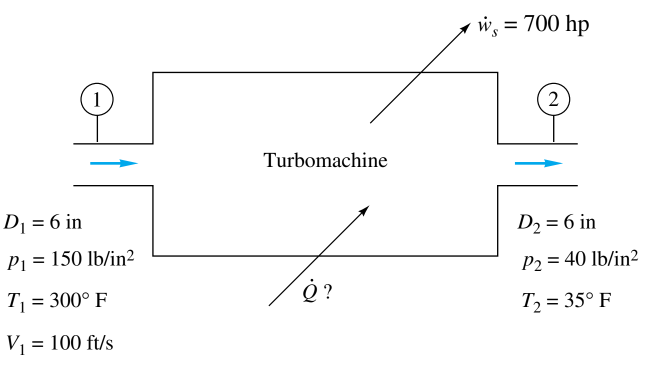
The friction head is larger than the elevation change Δz, and the pump must drive the flow against both changes, hence the high inlet pressure. The ratio of friction to velocity head is

$$\frac{h_f}{V^2/(2g)} \approx \frac{199\ \text{m}}{0.173\ \text{m}} \approx 1150 \qquad \textit{Ans.}$$

This high ratio is typical of long pipelines. (Note that we did not make direct use of the 10,000-m pipe length, whose effect is hidden within h_f.) In Chap. 6 we can state this problem in a more direct fashion: Given the flow rate, fluid, and pipe size, what inlet pressure is needed? Our correlations for h_f will lead to the estimate $p_{\text{inlet}} \approx 24$ atm, as stated above.

EXAMPLE 3.18

Air [$R = 1715$, $c_p = 6003$ ft · lbf/(slug · °R)] flows steadily, as shown in Fig. E3.18, through a turbine which produces 700 hp. For the inlet and exit conditions shown, estimate (*a*) the exit velocity V_2 and (*b*) the heat transferred $\dot{Q}$ in Btu/h.



E3.18

Solution

Part (a) The inlet and exit densities can be computed from the perfect-gas law:

$$\rho_1 = \frac{p_1}{RT_1} = \frac{150(144)}{1715(460 + 300)} = 0.0166 \text{ slug/ft}^3$$

$$\rho_2 = \frac{p_2}{RT_2} = \frac{40(144)}{1715(460 + 35)} = 0.00679 \text{ slug/ft}^3$$

The mass flow is determined by the inlet conditions

$$\dot{m} = \rho_1 A_1 V_1 = (0.0166)\frac{\pi}{4}\left(\frac{6}{12}\right)^2(100) = 0.325 \text{ slug/s}$$

Knowing mass flow, we compute the exit velocity

$$\dot{m} = 0.325 = \rho_2 A_2 V_2 = (0.00679)\frac{\pi}{4}\left(\frac{6}{12}\right)^2 V_2$$

or $$V_2 = 244 \text{ ft/s}$$ *Ans. (a)*

Part (b) The steady-flow energy equation (3.65) applies with $\dot{W}_v = 0$, $z_1 = z_2$, and $\hat{h} = c_p T$:

$$\dot{Q} - \dot{W}_s = \dot{m}(c_p T_2 + \tfrac{1}{2}V_2^2 - c_p T_1 - \tfrac{1}{2}V_1^2)$$

Convert the turbine work to foot-pounds-force per second with the conversion factor 1 hp = 550 ft · lbf/s. The turbine work is positive

$$\dot{Q} - 700(550) = 0.325[6003(495) + \tfrac{1}{2}(244)^2 - 6003(760) - \tfrac{1}{2}(100)^2]$$

$$= -510{,}000 \text{ ft} \cdot \text{lbf/s}$$

or $$\dot{Q} = -125{,}000 \text{ ft} \cdot \text{lbf/s}$$

Convert this to British thermal units as follows:

$$\dot{Q} = (-125{,}000 \text{ ft} \cdot \text{lbf/s})\frac{3600 \text{ s/h}}{778.2 \text{ ft} \cdot \text{lbf/Btu}}$$

$$= -576{,}000 \text{ Btu/h}$$ *Ans. (b)*

The negative sign indicates that this heat transfer is a *loss* from the control volume.

Kinetic-Energy Correction Factor

Often the flow entering or leaving a port is not strictly one-dimensional. In particular, the velocity may vary over the cross section, as in Fig. E3.4. In this case the kinetic-energy term in Eq. (3.64) for a given port should be modified by a dimensionless correction factor α so that the integral can be proportional to the square of the average velocity through the port

$$\int_{\text{port}} (\tfrac{1}{2}V^2)\rho(\mathbf{V} \cdot \mathbf{n})\, dA \equiv \alpha(\tfrac{1}{2}V_{av}^2)\dot{m}$$

where $$V_{av} = \frac{1}{A}\int u\, dA \qquad \text{for incompressible flow}$$

If the density is also variable, the integration is very cumbersome; we shall not treat this complication. By letting u be the velocity normal to the port, the first equation above becomes, for incompressible flow,

$$\tfrac{1}{2}\rho \int u^3 dA = \tfrac{1}{2}\rho\alpha V_{av}^3 A$$

or

$$\alpha = \frac{1}{A}\int \left(\frac{u}{V_{av}}\right)^3 dA \qquad (3.70)$$

The term α is the kinetic-energy correction factor, having a value of about 2.0 for fully developed laminar pipe flow and from 1.04 to 1.11 for turbulent pipe flow. The complete incompressible steady-flow energy equation (3.69), including pumps, turbines, and losses, would generalize to

$$\left(\frac{p}{\rho g} + \frac{\alpha}{2g}V^2 + z\right)_{in} = \left(\frac{p}{\rho g} + \frac{\alpha}{2g}V^2 + z\right)_{out} + h_{turbine} - h_{pump} + h_{friction} \qquad (3.71)$$

where the head terms on the right (h_t, h_p, h_f) are all numerically positive. All additive terms in Eq. (3.71) have dimensions of length $\{L\}$. In problems involving turbulent pipe flow, it is common to assume that $\alpha \approx 1.0$. To compute numerical values, we can use these approximations to be discussed in Chap. 6:

Laminar flow:
$$u = U_0\left[1 - \left(\frac{r}{R}\right)^2\right]$$

from which
$$V_{av} = 0.5U_0$$

and
$$\alpha = 2.0 \qquad (3.72)$$

Turbulent flow:
$$u \approx U_0\left(1 - \frac{r}{R}\right)^m \qquad m \approx \frac{1}{7}$$

from which, in Example 3.4,

$$V_{av} = \frac{2U_0}{(1+m)(2+m)}$$

Substituting into Eq. (3.70) gives

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)} \qquad (3.73)$$

and numerical values are as follows:

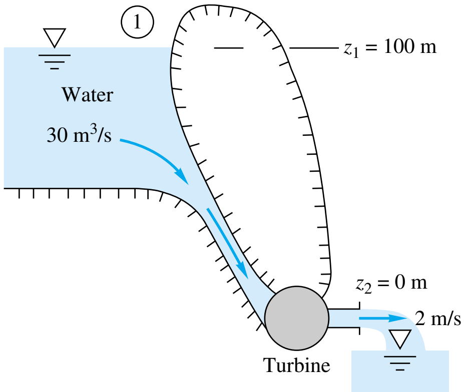
Turbulent flow:

m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
α	1.106	1.077	1.058	1.046	1.037

These values are only slightly different from unity and are often neglected in elementary turbulent-flow analyses. However, α should never be neglected in laminar flow.

EXAMPLE 3.19

A hydroelectric power plant (Fig. E3.19) takes in 30 m^3/s of water through its turbine and discharges it to the atmosphere at V_2 = 2 m/s. The head loss in the turbine and penstock system is h_f = 20 m. Assuming turbulent flow, $\alpha \approx 1.06$, estimate the power in MW extracted by the turbine.



E3.19

Solution

We neglect viscous work and heat transfer and take section 1 at the reservoir surface (Fig. E3.19), where $V_1 \approx 0$, $p_1 = p_{atm}$, and z_1 = 100 m. Section 2 is at the turbine outlet. The steady-flow energy equation (3.71) becomes, in head form,

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_t + h_f$$

$$\frac{p_a}{\gamma} + \frac{1.06(0)^2}{2(9.81)} + 100 \text{ m} = \frac{p_a}{\gamma} + \frac{1.06(2.0 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0 \text{ m} + h_t + 20 \text{ m}$$

The pressure terms cancel, and we may solve for the turbine head (which is positive):

$$h_t = 100 - 20 - 0.2 \approx 79.8 \text{ m}$$

The turbine extracts about 79.8 percent of the 100-m head available from the dam. The total power extracted may be evaluated from the water mass flow:

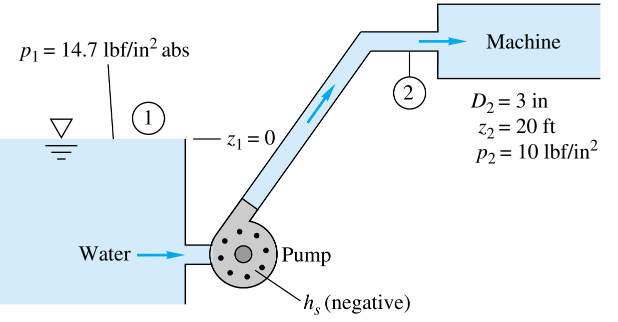
$$P = \dot{m} w_s = (\rho Q)(g h_t) = (998 \text{ kg/m}^3)(30 \text{ m}^3\text{/s})(9.81 \text{ m/s}^2)(79.8 \text{ m})$$

$$= 23.4 \text{ E6 kg} \cdot \text{m}^2/\text{s}^3 = 23.4 \text{ E6 N} \cdot \text{m/s} = 23.4 \text{ MW} \qquad \textit{Ans. 7}$$

The turbine drives an electric generator which probably has losses of about 15 percent, so the net power generated by this hydroelectric plant is about 20 MW.

EXAMPLE 3.20

The pump in Fig. E3.20 delivers water (62.4 lbf/ft^3) at 3 ft^3/s to a machine at section 2, which is 20 ft higher than the reservoir surface. The losses between 1 and 2 are given by $h_f = KV_2^2/(2g)$,



E3.20

where $K \approx 7.5$ is a dimensionless loss coefficient (see Sec. 6.7). Take $\alpha \approx 1.07$. Find the horsepower required for the pump if it is 80 percent efficient.

Solution

If the reservoir is large, the flow is steady, with $V_1 \approx 0$. We can compute V_2 from the given flow rate and the pipe diameter:

$$V_2 = \frac{Q}{A_2} = \frac{3 \text{ ft}^3/\text{s}}{(\pi/4)(\frac{3}{12} \text{ ft})^2} = 61.1 \text{ ft/s}$$

The viscous work is zero because of the solid walls and near-one-dimensional inlet and exit. The steady-flow energy equation (3.71) becomes

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_s + h_f$$

Introducing $V_1 \approx 0$, $z_1 = 0$, and $h_f = KV_2^2/(2g)$, we may solve for the pump head:

$$h_s = \frac{p_1 - p_2}{\gamma} - z_2 - (\alpha_2 + K)\left(\frac{V_2^2}{2g}\right)$$

The pressures should be in lbf/ft^2 for consistent units. For the given data, we obtain

$$h_s = \frac{(14.7 - 10.0)(144) \text{ lbf/ft}^2}{62.4 \text{ lbf/ft}^3} - 20 \text{ ft} - (1.07 + 7.5)\,\frac{(61.1 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)}$$

$$= 11 - 20 - 497 = -506 \text{ ft}$$

The pump head is negative, indicating work done *on* the fluid. As in Example 3.19, the power delivered is computed from

$$P = \dot{m}w_s = \rho Q g h_s = (1.94 \text{ slug/ft}^3)(3.0 \text{ ft}^3/\text{s})(32.2 \text{ ft/s}^2)(-507 \text{ ft}) = -94{,}900 \text{ ft} \cdot \text{lbf/s}$$

or

$$\text{hp} = \frac{94{,}900 \text{ ft} \cdot \text{lbf/s}}{550 \text{ ft} \cdot \text{lbf/(s} \cdot \text{hp)}} \approx 173 \text{ hp}$$

We drop the negative sign when merely referring to the "power" required. If the pump is 80 percent efficient, the input power required to drive it is

$$P_{\text{input}} = \frac{P}{\text{efficiency}} = \frac{173 \text{ hp}}{0.8} \approx 216 \text{ hp} \qquad \textit{Ans.}$$

The inclusion of the kinetic-energy correction factor α in this case made a difference of about 1 percent in the result.

3.7 Frictionless Flow: The Bernoulli Equation

Closely related to the steady-flow energy equation is a relation between pressure, velocity, and elevation in a frictionless flow, now called the *Bernoulli equation*. It was stated (vaguely) in words in 1738 in a textbook by Daniel Bernoulli. A complete derivation of the equation was given in 1755 by Leonhard Euler. The Bernoulli equation is very famous and very widely used, but one should be wary of its restrictions—all fluids are viscous and thus all flows have friction to some extent. To use the Bernoulli equation correctly, one must confine it to regions of the flow which are nearly frictionless. This section (and, in more detail, Chap. 8) will address the proper use of the Bernoulli relation.

Consider Fig. 3.15, which is an elemental fixed streamtube control volume of variable area $A(s)$ and length ds, where s is the streamline direction. The properties (ρ, V, p) may vary with s and time but are assumed to be uniform over the cross section A. The streamtube orientation θ is arbitrary, with an elevation change $dz = ds \sin\theta$. Friction on the streamtube walls is shown and then neglected—a very restrictive assumption.

Conservation of mass (3.20) for this elemental control volume yields

$$\frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \dot{m}_{\text{out}} - \dot{m}_{\text{in}} = 0 \approx \frac{\partial \rho}{\partial t} d\mathcal{V} + d\dot{m}$$

where $\dot{m} = \rho AV$ and $d\mathcal{V} \approx A\, ds$. Then our desired form of mass conservation is

$$d\dot{m} = d(\rho AV) = -\frac{\partial \rho}{\partial t} A\, ds \tag{3.74}$$

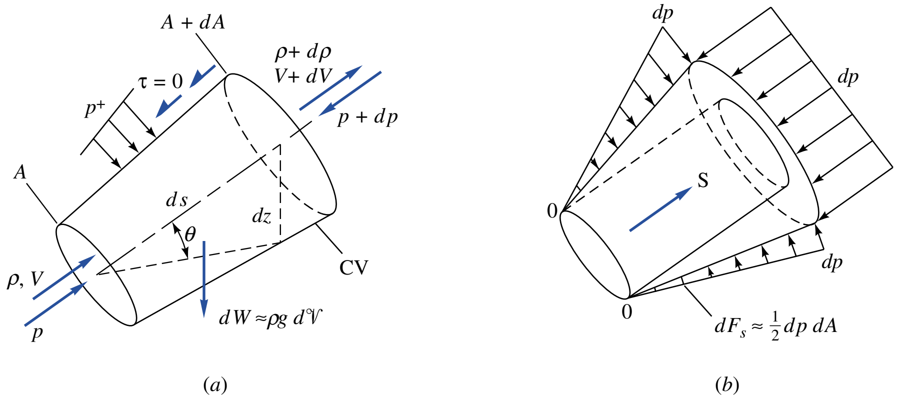


Fig. 3.15 The Bernoulli equation for frictionless flow along a streamline: (*a*) forces and fluxes; (*b*) net pressure force after uniform subtraction of p.

This relation does not require an assumption of frictionless flow.

Now write the linear-momentum relation (3.37) in the streamwise direction:

$$\sum dF_s = \frac{d}{dt}\left(\int_{CV} V\rho \, d\mathcal{V}\right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} \approx \frac{\partial}{\partial t}(\rho V)\, A\, ds + d(\dot{m}V)$$

where $V_s = V$ itself because s is the streamline direction. If we neglect the shear force on the walls (frictionless flow), the forces are due to pressure and gravity. The streamwise gravity force is due to the weight component of the fluid within the control volume:

$$dF_{s,grav} = -dW \sin\theta = -\gamma A\, ds \sin\theta = -\gamma A\, dz$$

The pressure force is more easily visualized, in Fig. 3.15*b*, by first subtracting a uniform value p from all surfaces, remembering from Fig. 3.7 that the net force is not changed. The pressure along the slanted side of the streamtube has a streamwise component which acts not on A itself but on the outer ring of area increase dA. The net pressure force is thus

$$dF_{s,press} = \tfrac{1}{2}\, dp\, dA - dp(A + dA) \approx -A\, dp$$

to first order. Substitute these two force terms into the linear-momentum relation:

$$\sum dF_s = -\gamma A\, dz - A\, dp = \frac{\partial}{\partial t}(\rho V)\, A\, ds + d(\dot{m}V)$$

$$= \frac{\partial \rho}{\partial t} VA\, ds + \frac{\partial V}{\partial t}\rho A\, ds + \dot{m}\, dV + V\, d\dot{m}$$

The first and last terms on the right cancel by virtue of the continuity relation (3.74). Divide what remains by ρA and rearrange into the final desired relation:

$$\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + V\, dV + g\, dz = 0 \qquad (3.75)$$

This is Bernoulli's equation for *unsteady frictionless flow along a streamline*. It is in differential form and can be integrated between any two points 1 and 2 on the streamline:

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0 \qquad (3.76)$$

To evaluate the two remaining integrals, one must estimate the unsteady effect $\partial V/\partial t$ and the variation of density with pressure. At this time we consider only steady ($\partial V/\partial t = 0$) incompressible (constant-density) flow, for which Eq. (3.76) becomes

$$\frac{p_2 - p_1}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

or

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 = \text{const} \qquad (3.77)$$

This is the Bernoulli equation for steady frictionless incompressible flow along a streamline.

Relation between the Bernoulli and Steady-Flow Energy Equations

Equation (3.77) is a widely used form of the Bernoulli equation for incompressible steady frictionless streamline flow. It is clearly related to the steady-flow energy equation for a streamtube (flow with one inlet and one outlet), from Eq. (3.66), which we state as follows:

$$\frac{p_1}{\rho} + \frac{\alpha_1 V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{\alpha_2 V_2^2}{2} + gz_2 + (\hat{u}_2 - \hat{u}_1 - q) + w_s + w_v \qquad (3.78)$$

This relation is much more general than the Bernoulli equation, because it allows for (1) friction, (2) heat transfer, (3) shaft work, and (4) viscous work (another frictional effect).

If we compare the Bernoulli equation (3.77) with the energy equation (3.78), we see that the Bernoulli equation contains even more restrictions than might first be realized. The complete list of assumptions for Eq. (3.77) is as follows:

1. *Steady flow*—a common assumption applicable to many flows.
2. *Incompressible flow*—acceptable if the flow Mach number is less than 0.3.
3. *Frictionless flow*—very restrictive, solid walls introduce friction effects.
4. *Flow along a single streamline*—different streamlines may have different "Bernoulli constants" $w_0 = p/\rho + V^2/2 + gz$, depending upon flow conditions.
5. *No shaft work between 1 and 2*—no pumps or turbines on the streamline.
6. *No heat transfer between 1 and 2*—either added or removed.

Thus our warning: Be wary of misuse of the Bernoulli equation. Only a certain limited set of flows satisfies all six assumptions above. The usual momentum or "mechanical force" derivation of the Bernoulli equation does not even reveal items 5 and 6, which are thermodynamic limitations. The basic reason for restrictions 5 and 6 is that heat transfer and work transfer, in real fluids, are married to frictional effects, which therefore invalidate our assumption of frictionless flow.

Figure 3.16 illustrates some practical limitations on the use of Bernoulli's equation (3.77). For the wind-tunnel model test of Fig. 3.16*a*, the Bernoulli equation is valid in the core flow of the tunnel but not in the tunnel-wall boundary layers, the model surface boundary layers, or the wake of the model, all of which are regions with high friction.

In the propeller flow of Fig. 3.16*b*, Bernoulli's equation is valid both upstream and downstream, but with a different constant $w_0 = p/\rho + V^2/2 + gz$, caused by the work addition of the propeller. The Bernoulli relation (3.77) is not valid near the propeller blades or in the helical vortices (not shown, see Fig. 1.12*a*) shed downstream of the blade edges. Also, the Bernoulli constants are higher in the flowing "slipstream" than in the ambient atmosphere because of the slipstream kinetic energy.

For the chimney flow of Fig. 3.16*c*, Eq. (3.77) is valid before and after the fire, but with a change in Bernoulli constant that is caused by heat addition. The Bernoulli equation is not valid within the fire itself or in the chimney-wall boundary layers.

The moral is to apply Eq. (3.77) only when all six restrictions can be satisfied: steady incompressible flow along a streamline with no friction losses, no heat transfer, and no shaft work between sections 1 and 2.

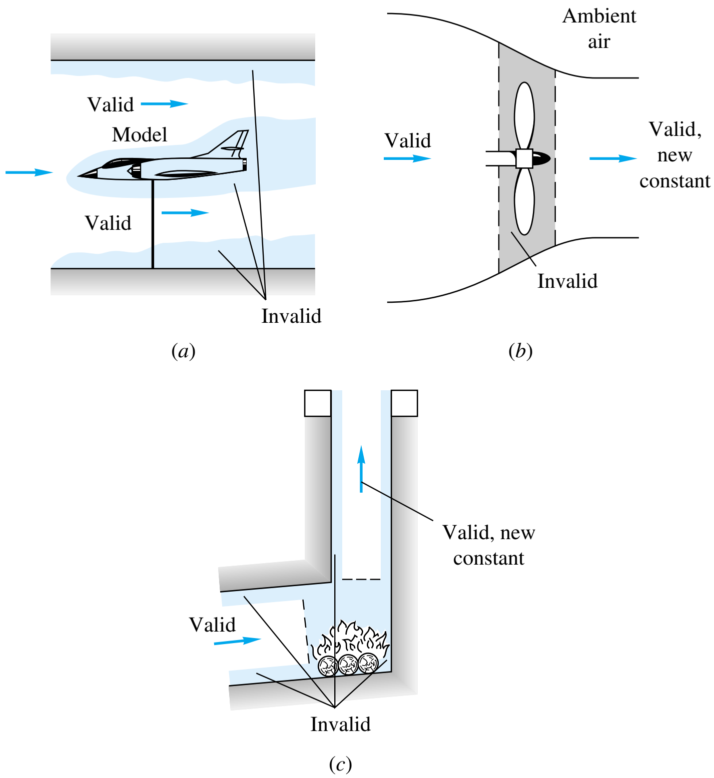


Fig. 3.16 Illustration of regions of validity and invalidity of the Bernoulli equation: (*a*) tunnel model, (*b*) propeller, (*c*) chimney.

Hydraulic and Energy Grade Lines

A useful visual interpretation of Bernoulli's equation is to sketch two grade lines of a flow. The *energy grade line* (EGL) shows the height of the total Bernoulli constant $h_0 = z + p/\gamma + V^2/(2g)$. In frictionless flow with no work or heat transfer, Eq. (3.77), the EGL has constant height. The *hydraulic grade line* (HGL) shows the height corresponding to elevation and pressure head $z + p/\gamma$, that is, the EGL minus the velocity head $V^2/(2g)$. The HGL is the height to which liquid would rise in a piezometer tube (see Prob. 2.11) attached to the flow. In an open-channel flow the HGL is identical to the free surface of the water.

Figure 3.17 illustrates the EGL and HGL for frictionless flow at sections 1 and 2 of a duct. The piezometer tubes measure the static-pressure head $z + p/\gamma$ and thus outline the HGL. The pitot stagnation-velocity tubes measure the total head $z + p/\gamma + V^2/(2g)$, which corresponds to the EGL. In this particular case the EGL is constant, and the HGL rises due to a drop in velocity.

In more general flow conditions, the EGL will drop slowly due to friction losses and will drop sharply due to a substantial loss (a valve or obstruction) or due to work extraction (to a turbine). The EGL can rise only if there is work addition (as from a pump or propeller). The HGL generally follows the behavior of the EGL with respect to losses or work transfer, and it rises and/or falls if the velocity decreases and/or increases.

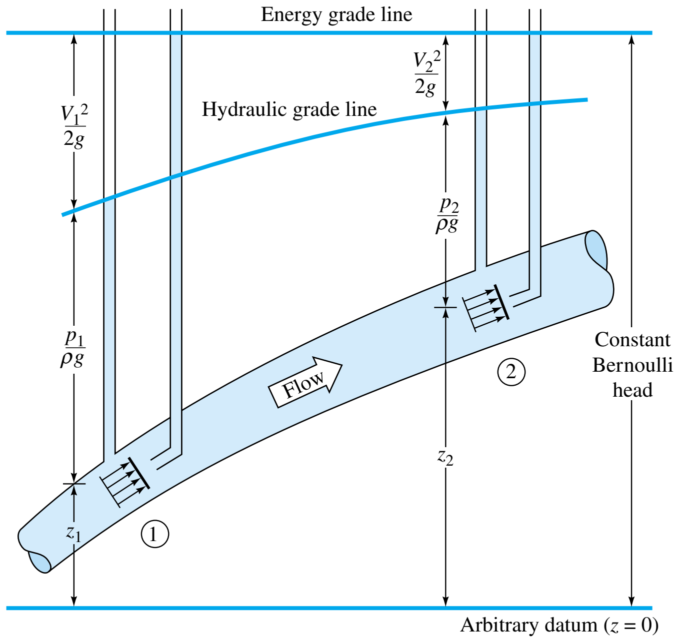


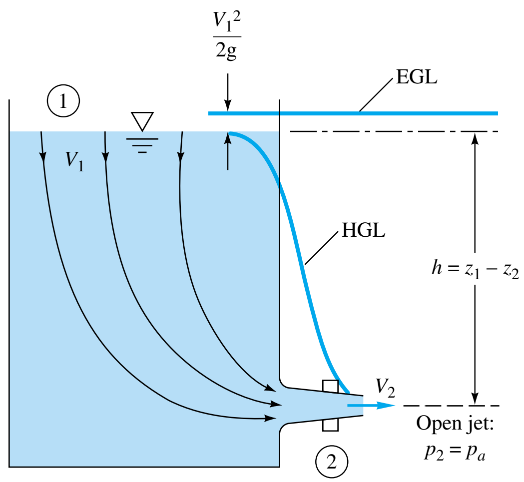
Fig. 3.17 Hydraulic and energy grade lines for frictionless flow in a duct.

As mentioned before, no conversion factors are needed in computations with the Bernoulli equation if consistent SI or BG units are used, as the following examples will show.

In all Bernoulli-type problems in this text, we consistently take point 1 upstream and point 2 downstream.

EXAMPLE 3.21

Find a relation between nozzle discharge velocity $V^2$and tank free-surface height h as in Fig. E3.21. Assume steady frictionless flow.



E3.21

Solution

As mentioned, we always choose point 1 upstream and point 2 downstream. Try to choose points 1 and 2 where maximum information is known or desired. Here we select point 1 as the tank free surface, where elevation and pressure are known, and point 2 as the nozzle exit, where again pressure and elevation are known. The two unknowns are V_1 and V_2.

Mass conservation is usually a vital part of Bernoulli analyses. If A_1 is the tank cross section and A_2 the nozzle area, this is approximately a one-dimensional flow with constant density, Eq. (3.30),

$$A_1 V_1 = A_2 V_2 \tag{1}$$

Bernoulli's equation (3.77) gives

$$\frac{p_1}{\rho} + \tfrac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \tfrac{1}{2}V_2^2 + gz_2$$

But since sections 1 and 2 are both exposed to atmospheric pressure $p_1 = p_2 = p_a$, the pressure terms cancel, leaving

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh \tag{2}$$

Eliminating V_1 between Eqs. (1) and (2), we obtain the desired result:

$$V_2^2 = \frac{2gh}{1 - A_2^2/A_1^2} \qquad \textit{Ans.} \ (3)$$

Generally the nozzle area A_2 is very much smaller than the tank area A_1, so that the ratio A_2^2/A_1^2 is doubly negligible, and an accurate approximation for the outlet velocity is

$$V_2 \approx (2gh)^{1/2} \qquad \textit{Ans.} \ (4)$$

This formula, discovered by Evangelista Torricelli in 1644, states that the discharge velocity equals the speed which a frictionless particle would attain if it fell freely from point 1 to point 2. In other words, the potential energy of the surface fluid is entirely converted to kinetic energy of efflux, which is consistent with the neglect of friction and the fact that no net pressure work is done. Note that Eq. (4) is independent of the fluid density, a characteristic of gravity-driven flows.

Except for the wall boundary layers, the streamlines from 1 to 2 all behave in the same way, and we can assume that the Bernoulli constant h_0 is the same for all the core flow. However, the outlet flow is likely to be nonuniform, not one-dimensional, so that the average velocity is only approximately equal to Torricelli's result. The engineer will then adjust the formula to include a dimensionless *discharge coefficient* c_d

$$(V_2)_{\text{av}} = \frac{Q}{A_2} = c_d(2gh)^{1/2} \tag{5}$$

As discussed in Sec. 6.10, the discharge coefficient of a nozzle varies from about 0.6 to 1.0 as a function of (dimensionless) flow conditions and nozzle shape.

Before proceeding with more examples, we should note carefully that a solution by Bernoulli's equation (3.77) does *not* require a control-volume analysis, only a selection of two points 1 and 2 along a given streamline. The control volume was used to derive the differential relation (3.75), but the integrated form (3.77) is valid all along the streamline for frictionless flow with no heat transfer or shaft work, and a control volume is not necessary.

EXAMPLE 3.22

Rework Example 3.21 to account, at least approximately, for the unsteady-flow condition caused by the draining of the tank.

Solution

Essentially we are asked to include the unsteady integral term involving $\partial V/\partial t$ from Eq. (3.76). This will result in a new term added to Eq. (2) from Example 3.21:

$$2\int_1^2 \frac{\partial V}{\partial t}\,ds + V_2^2 - V_1^2 = 2gh \tag{1}$$

Since the flow is incompressible, the continuity equation still retains the simple form $A_1V_1 = A_2V_2$ from Example 3.21. To integrate the unsteady term, we must estimate the acceleration all along the streamline. Most of the streamline is in the tank region where $\partial V/\partial t \approx dV_1/dt$. The length of the average streamline is slightly longer than the nozzle depth h. A crude estimate for the integral is thus

$$\int_1^2 \frac{\partial V}{\partial t}\,ds \approx \int_1^2 \frac{dV_1}{dt}\,ds \approx -\frac{dV_1}{dt}\,h \tag{2}$$

But since A_1 and A_2 are constant, $dV_1/dt \approx (A_2/A_1)(dV_2/dt)$. Substitution into Eq. (1) gives

$$-2h\,\frac{A_2}{A_1}\,\frac{dV_2}{dt} + V_2^2\left(1 - \frac{A_2^2}{A_1^2}\right) \approx 2gh \tag{3}$$

This is a first-order differential equation for $V_2(t)$. It is complicated by the fact that the depth h is variable; therefore $h = h(t)$, as determined by the variation in $V_1(t)$

$$h(t) = h_0 - \int_0^t V_1\,dt \tag{4}$$

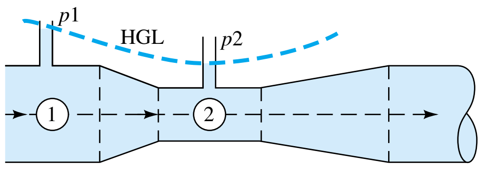
Equations (3) and (4) must be solved simultaneously, but the problem is well posed and can be handled analytically or numerically. We can also estimate the size of the first term in Eq. (3) by using the approximation $V_2 \approx (2gh)^{1/2}$ from the previous example. After differentiation, we obtain

$$2h\,\frac{A_2}{A_1}\,\frac{dV_2}{dt} \approx -\left(\frac{A_2}{A_1}\right)^2 V_2^2 \tag{5}$$

which is negligible if $A_2 \ll A_1$, as originally postulated.

EXAMPLE 3.23

A constriction in a pipe will cause the velocity to rise and the pressure to fall at section 2 in the throat. The pressure difference is a measure of the flow rate through the pipe. The smoothly necked-down system shown in Fig. E3.23 is called a *venturi tube*. Find an expression for the mass flux in the tube as a function of the pressure change.



E3.23

Solution

Bernoulli's equation is assumed to hold along the center streamline

$$\frac{p_1}{\rho} + \tfrac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \tfrac{1}{2}V_2^2 + gz_2$$

If the tube is horizontal, $z_1 = z_2$ and we can solve for V_2:

$$V_2^2 - V_1^2 = \frac{2\,\Delta p}{\rho} \qquad \Delta p = p_1 - p_2 \tag{1}$$

We relate the velocities from the incompressible continuity relation

$$A_1V_1 = A_2V_2$$

or

$$V_1 = \beta^2 V_2 \qquad \beta = \frac{D_2}{D_1} \tag{2}$$

Combining (1) and (2), we obtain a formula for the velocity in the throat

$$V_2 = \left[\frac{2\,\Delta p}{\rho(1-\beta^4)}\right]^{1/2} \tag{3}$$

The mass flux is given by

$$\dot m = \rho A_2 V_2 = A_2\left(\frac{2\rho\,\Delta p}{1-\beta^4}\right)^{1/2} \tag{4}$$

This is the ideal frictionless mass flux. In practice, we measure $\dot m_{\text{actual}} = c_d \dot m_{\text{ideal}}$ and correlate the discharge coefficient c_d.

EXAMPLE 3.24

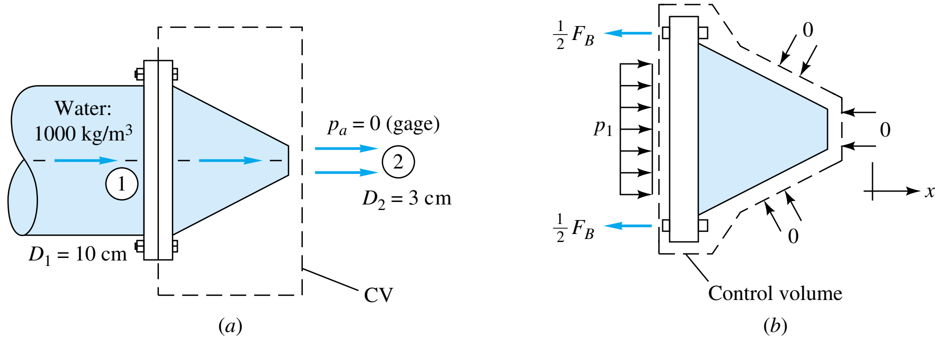
A 10-cm fire hose with a 3-cm nozzle discharges 1.5 m^3/min to the atmosphere. Assuming frictionless flow, find the force F_B exerted by the flange bolts to hold the nozzle on the hose.

Solution

We use Bernoulli's equation and continuity to find the pressure p_1 upstream of the nozzle and then we use a control-volume momentum analysis to compute the bolt force, as in Fig. E3.24.

The flow from 1 to 2 is a constriction exactly similar in effect to the venturi in Example 3.23 for which Eq. (1) gave

$$p_1 = p_2 + \tfrac{1}{2}\rho(V_2^2 - V_1^2) \tag{1}$$



E3.24

The velocities are found from the known flow rate $Q = 1.5\ \text{m}^3/\text{min}$ or $0.025\ \text{m}^3/\text{s}$:

$$V_2 = \frac{Q}{A_2} = \frac{0.025\ \text{m}^3/\text{s}}{(\pi/4)(0.03\ \text{m})^2} = 35.4\ \text{m/s}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.025\ \text{m}^3/\text{s}}{(\pi/4)(0.1\ \text{m})^2} = 3.2\ \text{m/s}$$

We are given $p_2 = p_a = 0$ gage pressure. Then Eq. (1) becomes

$$p_1 = \tfrac{1}{2}(1000\ \text{kg/m}^3)[(35.4^2 - 3.2^2)\ \text{m}^2/\text{s}^2]$$

$$= 620{,}000\ \text{kg/(m}\cdot\text{s}^2) = 620{,}000\ \text{Pa gage}$$

The control-volume force balance is shown in Fig. E3.24*b*:

$$\sum F_x = -F_B + p_1 A_1$$

and the zero gage pressure on all other surfaces contributes no force. The x-momentum flux is $+\dot{m}V_2$ at the outlet and $-\dot{m}V_1$ at the inlet. The steady-flow momentum relation (3.40) thus gives

$$-F_B + p_1 A_1 = \dot{m}(V_2 - V_1)$$

or

$$F_B = p_1 A_1 - \dot{m}(V_2 - V_1) \qquad (2)$$

Substituting the given numerical values, we find

$$\dot{m} = \rho Q = (1000\ \text{kg/m}^3)(0.025\ \text{m}^3/\text{s}) = 25\ \text{kg/s}$$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4}(0.1\ \text{m})^2 = 0.00785\ \text{m}^2$$

$$F_B = (620{,}000\ \text{N/m}^2)(0.00785\ \text{m}^2) - (25\ \text{kg/s})[(35.4 - 3.2)\ \text{m/s}]$$

$$= 4872\ \text{N} - 805\ (\text{kg}\cdot\text{m})/\text{s}^2 = 4067\ \text{N}\ (915\ \text{lbf}) \qquad \textit{Ans.}$$

This gives an idea of why it takes more than one firefighter to hold a fire hose at full discharge.

Notice from these examples that the solution of a typical problem involving Bernoulli's equation almost always leads to a consideration of the continuity equation as an equal partner in the analysis. The only exception is when the complete velocity distribution is already known from a previous or given analysis, but that means that the continuity relation has already been used to obtain the given information. The point is that the continuity relation is always an important element in a flow analysis.

Summary

This chapter has analyzed the four basic equations of fluid mechanics: conservation of (1) mass, (2) linear momentum, (3) angular momentum, and (4) energy. The equations were attacked "in the large," i.e., applied to whole regions of a flow. As such, the typical analysis will involve an approximation of the flow field within the region, giving somewhat crude but always instructive quantitative results. However, the basic control-volume relations are rigorous and correct and will give exact results if applied to the exact flow field.

There are two main points to a control-volume analysis. The first is the selection of a proper, clever, workable control volume. There is no substitute for experience, but the following guidelines apply. The control volume should cut through the place where the information or solution is desired. It should cut through places where maximum information is already known. If the momentum equation is to be used, it should *not* cut through solid walls unless absolutely necessary, since this will expose possible unknown stresses and forces and moments which make the solution for the desired force difficult or impossible. Finally, every attempt should be made to place the control volume in a frame of reference where the flow is steady or quasi-steady, since the steady formulation is much simpler to evaluate.

The second main point to a control-volume analysis is the reduction of the analysis to a case which applies to the problem at hand. The 24 examples in this chapter give only an introduction to the search for appropriate simplifying assumptions. You will need to solve 24 or 124 more examples to become truly experienced in simplifying the problem just enough and no more. In the meantime, it would be wise for the beginner to adopt a very general form of the control-volume conservation laws and then make a series of simplifications to achieve the final analysis. Starting with the general form, one can ask a series of questions:

1. Is the control volume nondeforming or nonaccelerating?
2. Is the flow field steady? Can we change to a steady-flow frame?
3. Can friction be neglected?
4. Is the fluid incompressible? If not, is the perfect-gas law applicable?
5. Are gravity or other body forces negligible?
6. Is there heat transfer, shaft work, or viscous work?
7. Are the inlet and outlet flows approximately one-dimensional?
8. Is atmospheric pressure important to the analysis? Is the pressure hydrostatic on any portions of the control surface?
9. Are there reservoir conditions which change so slowly that the velocity and time rates of change can be neglected?

In this way, by approving or rejecting a list of basic simplifications like those above, one can avoid pulling Bernoulli's equation off the shelf when it does not apply.