# Chapter 5
# Dimensional Analysis and Similarity

**Motivation.** In this chapter we discuss the planning, presentation, and interpretation of experimental data. We shall try to convince you that such data are best presented in *dimensionless* form. Experiments which might result in tables of output, or even multiple volumes of tables, might be reduced to a single set of curves—or even a single curve—when suitably nondimensionalized. The technique for doing this is *dimensional analysis*.

Chapter 3 presented gross control-volume balances of mass, momentum, and energy which led to estimates of global parameters: mass flow, force, torque, total heat transfer. Chapter 4 presented infinitesimal balances which led to the basic partial differential equations of fluid flow and some particular solutions. These two chapters covered *analytical* techniques, which are limited to fairly simple geometries and well-defined boundary conditions. Probably one-third of fluid-flow problems can be attacked in this analytical or theoretical manner.

The other two-thirds of all fluid problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiment. Their behavior is reported as experimental data. Such data are much more useful if they are expressed in compact, economic form. Graphs are especially useful, since tabulated data cannot be absorbed, nor can the trends and rates of change be observed, by most engineering eyes. These are the motivations for dimensional analysis. The technique is traditional in fluid mechanics and is useful in all engineering and physical sciences, with notable uses also seen in the biological and social sciences.

Dimensional analysis can also be useful in theories, as a compact way to present an analytical solution or output from a computer model. Here we concentrate on the presentation of experimental fluid-mechanics data.

## 5.1 Introduction

Basically, dimensional analysis is a method for reducing the number and complexity of experimental variables which affect a given physical phenomenon, by using a sort of compacting technique. If a phenomenon depends upon $n$ dimensional variables, dimensional analysis will reduce the problem to only $k$ *dimensionless* variables, where the reduction $n - k = 1, 2, 3$, or 4, depending upon the problem complexity. Generally $n - k$ equals the number of different dimensions (sometimes called basic or primary or fundamental dimensions) which govern the problem. In fluid mechanics, the four basic dimensions are usually taken to be mass $M$, length $L$, time $T$, and temperature $\Theta$, or an $MLT\Theta$ system for short. Sometimes one uses an $FLT\Theta$ system, with force $F$ replacing mass.

Although its purpose is to reduce variables and group them in dimensionless form, dimensional analysis has several side benefits. The first is enormous savings in time and money. Suppose one knew that the force $F$ on a particular body immersed in a stream of fluid depended only on the body length $L$, stream velocity $V$, fluid density $\rho$, and fluid viscosity $\mu$, that is,

$$F = f(L, V, \rho, \mu) \tag{5.1}$$

Suppose further that the geometry and flow conditions are so complicated that our integral theories (Chap. 3) and differential equations (Chap. 4) fail to yield the solution for the force. Then we must find the function $f(L, V, \rho, \mu)$ experimentally.

Generally speaking, it takes about 10 experimental points to define a curve. To find the effect of body length in Eq. (5.1), we have to run the experiment for 10 lengths $L$. For each $L$ we need 10 values of $V$, 10 values of $\rho$, and 10 values of $\mu$, making a grand total of $10^4$, or 10,000, experiments. At \$50 per experiment—well, you see what we are getting into. However, with dimensional analysis, we can immediately reduce Eq. (5.1) to the equivalent form

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \tag{5.2}$$

or

$$C_F = g(\text{Re})$$

i.e., the dimensionless *force coefficient* $F/(\rho V^2 L^2)$ is a function only of the dimensionless *Reynolds number* $\rho VL/\mu$. We shall learn exactly how to make this reduction in Secs. 5.2 and 5.3.

The function $g$ is different mathematically from the original function $f$, but it contains all the same information. Nothing is lost in a dimensional analysis. And think of the savings: We can establish $g$ by running the experiment for only 10 values of the single variable called the Reynolds number. We do not have to vary $L$, $V$, $\rho$, or $\mu$ separately but only the *grouping* $\rho VL/\mu$. This we do merely by varying velocity $V$ in, say, a wind tunnel or drop test or water channel, and there is no need to build 10 different bodies or find 100 different fluids with 10 densities and 10 viscosities. The cost is now about \$500, maybe less.

A second side benefit of dimensional analysis is that it helps our thinking and planning for an experiment or theory. It suggests dimensionless ways of writing equations before we waste money on computer time to find solutions. It suggests variables which can be discarded; sometimes dimensional analysis will immediately reject variables, and at other times it groups them off to the side, where a few simple tests will show them to be unimportant. Finally, dimensional analysis will often give a great deal of insight into the form of the physical relationship we are trying to study.

A third benefit is that dimensional analysis provides *scaling laws* which can convert data from a cheap, small *model* to design information for an expensive, large *prototype*. We do not build a million-dollar airplane and see whether it has enough lift force. We measure the lift on a small model and use a scaling law to predict the lift on the full-scale prototype airplane. There are rules we shall explain for finding scaling laws. When the scaling law is valid, we say that a condition of *similarity* exists between the model and the prototype. In the simple case of Eq. (5.1), similarity is achieved if the Reynolds number is the same for the model and prototype because the function $g$ then requires the force coefficient to be the same also:

$$\text{If} \quad \text{Re}_m = \text{Re}_p \quad \text{then} \quad C_{Fm} = C_{Fp} \tag{5.3}$$

where subscripts $m$ and $p$ mean model and prototype, respectively. From the definition of force coefficient, this means that

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 \tag{5.4}$$

for data taken where $\rho_p V_p L_p/\mu_p = \rho_m V_m L_m/\mu_m$. Equation (5.4) is a scaling law: If you measure the model force at the model Reynolds number, the prototype force at the same Reynolds number equals the model force times the density ratio times the velocity ratio squared times the length ratio squared. We shall give more examples later.

Do you understand these introductory explanations? Be careful; learning dimensional analysis is like learning to play tennis: There are levels of the game. We can establish some ground rules and do some fairly good work in this brief chapter, but dimensional analysis in the broad view has many subtleties and nuances which only time and practice and maturity enable you to master. Although dimensional analysis has a firm physical and mathematical foundation, considerable art and skill are needed to use it effectively.

## EXAMPLE 5.1

A copepod is a water crustacean approximately 1 mm in diameter. We want to know the drag force on the copepod when it moves slowly in fresh water. A scale model 100 times larger is made and tested in glycerin at $V$ = 30 cm/s. The measured drag on the model is 1.3 N. For similar conditions, what are the velocity and drag of the actual copepod in water? Assume that Eq. (5.1) applies and the temperature is 20°C.

### Solution

From Table A.3 the fluid properties are:

| | | |
|---|---|---|
| Water (prototype): | $\mu_p = 0.001 \text{ kg/(m}\cdot\text{s)}$ | $\rho_p = 998 \text{ kg/m}^3$ |
| Glycerin (model): | $\mu_m = 1.5 \text{ kg/(m}\cdot\text{s)}$ | $\rho_m = 1263 \text{ kg/m}^3$ |

The length scales are $L_m$ = 100 mm and $L_p$ = 1 mm. We are given enough model data to compute the Reynolds number and force coefficient

$$\text{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \frac{(1263 \text{ kg/m}^3)(0.3 \text{ m/s})(0.1 \text{ m})}{1.5 \text{ kg/(m}\cdot\text{s)}} = 25.3$$

$$C_{Fm} = \frac{F_m}{\rho_m V_m{}^2 L_m{}^2} = \frac{1.3 \text{ N}}{(1263 \text{ kg/m}^3)(0.3 \text{ m/s})^2(0.1 \text{ m})^2} = 1.14$$

Both these numbers are dimensionless, as you can check. For conditions of similarity, the prototype Reynolds number must be the same, and Eq. (5.2) then requires the prototype force coefficient to be the same

$$\text{Re}_p = \text{Re}_m = 25.3 = \frac{998V_p(0.001)}{0.001}$$

or
$$V_p = 0.0253 \text{ m/s} = 2.53 \text{ cm/s} \qquad \textit{Ans.}$$

$$C_{Fp} = C_{Fm} = 1.14 = \frac{F_p}{998(0.0253)^2(0.001)^2}$$

or
$$F_p = 7.31 \times 10^{-7} \text{ N} \qquad \textit{Ans.}$$

It would obviously be difficult to measure such a tiny drag force.

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Historically, the first person to write extensively about units and dimensional reasoning in physical relations was Euler in 1765. Euler's ideas were far ahead of his time, as were those of Joseph Fourier, whose 1822 book *Analytical Theory of Heat* outlined what is now called the *principle of dimensional homogeneity* and even developed some similarity rules for heat flow. There were no further significant advances until Lord Rayleigh's book in 1877, *Theory of Sound,* which proposed a "method of dimensions" and gave several examples of dimensional analysis. The final breakthrough which established the method as we know it today is generally credited to E. Buckingham in 1914 [29], whose paper outlined what is now called the *Buckingham pi theorem* for describing dimensionless parameters (see Sec. 5.3). However, it is now known that a Frenchman, A. Vaschy, in 1892 and a Russian, D. Riabouchinsky, in 1911 had independently published papers reporting results equivalent to the pi theorem. Following Buckingham's paper, P. W. Bridgman published a classic book in 1922 [1], outlining the general theory of dimensional analysis. The subject continues to be controversial because there is so much art and subtlety in using dimensional analysis. Thus, since Bridgman there have been at least 24 books published on the subject [2 to 25]. There will probably be more, but seeing the whole list might make some fledgling authors think twice. Nor is dimensional analysis limited to fluid mechanics or even engineering. Specialized books have been written on the application of dimensional analysis to metrology [26], astrophysics [27], economics [28], building scale models [36], chemical processing pilot plants [37], social sciences [38], biomedical sciences [39], pharmacy [40], fractal geometry [41], and even the growth of plants [42].

## 5.2 The Principle of Dimensional Homogeneity

In making the remarkable jump from the five-variable Eq. (5.1) to the two-variable Eq. (5.2), we were exploiting a rule which is almost a self-evident axiom in physics. This rule, the *principle of dimensional homogeneity* (PDH), can be stated as follows:

> If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous;* i.e., each of its additive terms will have the same dimensions.

All the equations which are derived from the theory of mechanics are of this form. For example, consider the relation which expresses the displacement of a falling body

$$S = S_0 + V_0 t + \tfrac{1}{2}gt^2 \tag{5.5}$$

Each term in this equation is a displacement, or length, and has dimensions $\{L\}$. The equation is dimensionally homogeneous. Note also that any consistent set of units can be used to calculate a result.

Consider Bernoulli's equation for incompressible flow

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{const} \tag{5.6}$$

Each term, including the constant, has dimensions of velocity squared, or $\{L^2T^{-2}\}$. The equation is dimensionally homogeneous and gives proper results for any consistent set of units.

Students count on dimensional homogeneity and use it to check themselves when they cannot quite remember an equation during an exam. For example, which is it:

$$S = \tfrac{1}{2}gt^2? \qquad \text{or} \qquad S = \tfrac{1}{2}g^2t? \tag{5.7}$$

By checking the dimensions, we reject the second form and back up our faulty memory. We are exploiting the principle of dimensional homogeneity, and this chapter simply exploits it further.

Equations (5.5) and (5.6) also illustrate some other factors that often enter into a dimensional analysis:

*Dimensional variables* are the quantities which actually vary during a given case and would be plotted against each other to show the data. In Eq. (5.5), they are $S$ and $t$; in Eq. (5.6) they are $p$, $V$, and $z$. All have dimensions, and all can be nondimensionalized as a dimensional-analysis technique.

*Dimensional constants* may vary from case to case but are held constant during a given run. In Eq. (5.5) they are $S_0$, $V_0$, and $g$, and in Eq. (5.6) they are $\rho$, $g$, and $C$. They all have dimensions and conceivably could be nondimensionalized, but they are normally used to help nondimensionalize the variables in the problem.

*Pure constants* have no dimensions and never did. They arise from mathematical manipulations. In both Eqs. (5.5) and (5.6) they are $\frac{1}{2}$ and the exponent 2, both of which came from an integration: $\int t\,dt = \frac{1}{2}t^2$, $\int V\,dV = \frac{1}{2}V^2$. Other common dimensionless constants are $\pi$ and $e$.

Note that integration and differentiation of an equation may change the dimensions but not the homogeneity of the equation. For example, integrate or differentiate Eq. (5.5):

$$\int S\,dt = S_0t + \tfrac{1}{2}V_0t^2 + \tfrac{1}{6}gt^3 \tag{5.8$a$}$$

$$\frac{dS}{dt} = V_0 + gt \tag{5.8$b$}$$

In the integrated form (5.8*a*) every term has dimensions of $\{LT\}$, while in the derivative form (5.8*b*) every term is a velocity $\{LT^{-1}\}$.

Finally, there are some physical variables that are naturally dimensionless by virtue of their definition as ratios of dimensional quantities. Some examples are strain (change in length per unit length), Poisson's ratio (ratio of transverse strain to longitudinal strain), and specific gravity (ratio of density to standard water density). All angles are dimensionless (ratio of arc length to radius) and should be taken in radians for this reason.

The motive behind dimensional analysis is that any dimensionally homogeneous equation can be written in an entirely equivalent nondimensional form which is more compact. Usually there is more than one method of presenting one's dimensionless data or theory. Let us illustrate these concepts more thoroughly by using the falling-body relation (5.5) as an example.

## Ambiguity: The Choice of Variables and Scaling Parameters[1]

Equation (5.5) is familiar and simple, yet illustrates most of the concepts of dimensional analysis. It contains five terms ($S$, $S_0$, $V_0$, $t$, $g$) which we may divide, in our thinking, into variables and parameters. The *variables* are the things which we wish to plot, the basic output of the experiment or theory: in this case, $S$ versus $t$. The *parameters* are those quantities whose effect upon the variables we wish to know: in this case $S_0$, $V_0$, and $g$. Almost any engineering study can be subdivided in this manner.

To nondimensionalize our results, we need to know how many dimensions are contained among our variables and parameters: in this case, only two, length $\{L\}$ and time $\{T\}$. Check each term to verify this:

$$\{S\} = \{S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\}$$

Among our parameters, we therefore select two to be *scaling parameters,* used to define dimensionless variables. What remains will be the "basic" parameter(s) whose effect we wish to show in our plot. These choices will not affect the content of our data, only the form of their presentation. Clearly there is ambiguity in these choices, something that often vexes the beginning experimenter. But the ambiguity is deliberate. Its purpose is to show a particular effect, and the choice is yours to make.

For the falling-body problem, we select any two of the three parameters to be scaling parameters. Thus we have three options. Let us discuss and display them in turn.

***Option 1:*** Scaling parameters $S_0$ and $V_0$: the effect of gravity $g$.

First use the scaling parameters ($S_0$, $V_0$) to define dimensionless (*) displacement and time. There is only one suitable definition for each:[2]

$$S^* = \frac{S}{S_0} \qquad t^* = \frac{V_0 t}{S_0} \tag{5.9}$$

Substitute these variables into Eq. (5.5) and clean everything up until each term is dimensionless. The result is our first option:

$$S^* = 1 + t^* + \frac{1}{2}\alpha t^{*2} \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.10}$$

This result is shown plotted in Fig. 5.1*a*. There is a single dimensionless parameter $\alpha$, which shows here the effect of gravity. It cannot show the direct effects of $S_0$ and $V_0$, since these two are hidden in the ordinate and abscissa. We see that gravity increases the parabolic rate of fall for $t^* > 0$, but not the initial slope at $t^* = 0$. We would learn the same from falling-body data, and the plot, within experimental accuracy, would look like Fig. 5.1*a*.

[1] I am indebted to Prof. Jacques Lewalle of Syracuse University for suggesting, outlining, and clarifying this entire discussion.

[2] Make them *proportional* to $S$ and $t$. Do not define dimensionless terms upside down: $S_0/S$ or $S_0/(V_0 t)$. The plots will look funny, users of your data will be confused, and your supervisor will be angry. It is not a good idea.

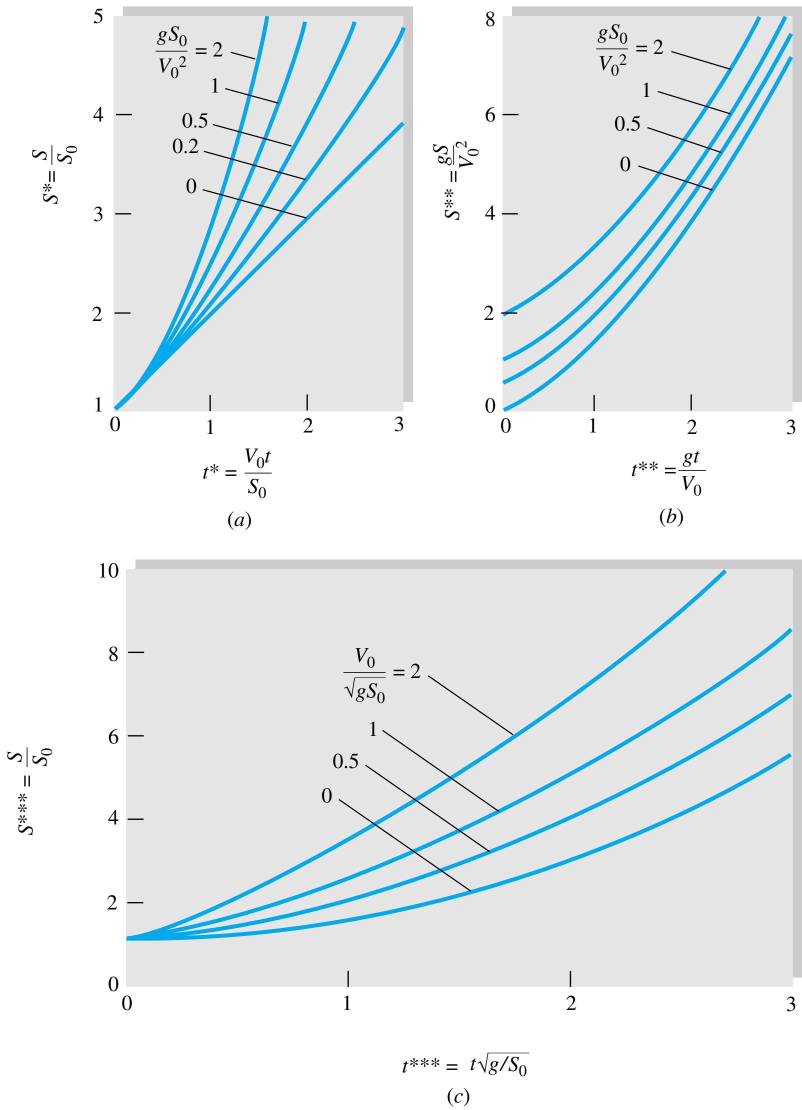



**Fig. 5.1** Three entirely equivalent dimensionless presentations of the falling-body problem, Eq. (5.5): the effect of (*a*) gravity, (*b*) initial displacement, and (*c*) initial velocity. All plots contain the same information.

***Option 2:*** Scaling parameters $V_0$ and $g$: the effect of initial displacement $S_0$.

Now use the new scaling parameters ($V_0$, $g$) to define dimensionless (**) displacement and time. Again there is only one suitable definition:

$$S^{**} = \frac{Sg}{V_0^2} \qquad t^{**} = t\frac{g}{V_0} \tag{5.11}$$

Substitute these variables into Eq. (5.5) and clean everything up again. The result is our second option:

$$S^{**} = \alpha + t^{**} + \frac{1}{2}t^{**2} \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.12}$$

This result is plotted in Fig. 5.1*b*. The same single parameter $\alpha$ again appears and here shows the effect of initial *displacement,* which merely moves the curves upward without changing their shape.

***Option 3:*** Scaling parameters $S_0$ and $g$: the effect of initial speed $V_0$.

Finally use the scaling parameters ($S_0$, $g$) to define dimensionless (***) displacement and time. Again there is only one suitable definition:

$$S^{***} = \frac{S}{S_0} \qquad t^{***} = t\left(\frac{g}{S_0}\right)^{1/2} \tag{5.13}$$

Substitute these variables into Eq. (5.5) and clean everything up as usual. The result is our third and final option:

$$S^{***} = 1 + \beta t^{***} + \frac{1}{2}t^{***2} \qquad \beta = \frac{1}{\sqrt{\alpha}} = \frac{V_0}{\sqrt{gS_0}} \tag{5.14}$$

This final presentation is shown in Fig. 5.1*c*. Once again the parameter $\alpha$ appears, but we have redefined it upside down, $\beta = 1/\sqrt{\alpha}$, so that our display parameter $V_0$ is in the numerator and is linear. This is our free choice and simply improves the display. Figure 5.1*c* shows that initial *velocity* increases the falling displacement and that the increase is proportional to time.

Note that, in all three options, the same parameter $\alpha$ appears but has a different meaning: dimensionless gravity, initial displacement, and initial velocity. The graphs, which contain exactly the same information, change their appearance to reflect these differences.

Whereas the original problem, Eq. (5.5), involved five quantities, the dimensionless presentations involve only three, having the form

$$S' = \text{fcn}(t', \alpha) \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.15}$$

The reduction $5 - 3 = 2$ should equal the number of fundamental dimensions involved in the problem $\{L, T\}$. This idea led to the pi theorem (Sec. 5.3).

The choice of scaling variables is left to the user, and the resulting dimensionless parameters have differing interpretations. For example, in the dimensionless drag-force formulation, Eq. (5.2), it is now clear that the scaling parameters were $\rho$, $V$, and $L$, since they appear in both the drag coefficient and the Reynolds number. Equation (5.2) can thus be interpreted as the variation of dimensionless *force* with dimensionless *viscosity,* with the scaling-parameter effects mixed between $C_F$ and Re and therefore not immediately evident.

Suppose that we wish to study drag force versus *velocity*. Then we would not use $V$ as a scaling parameter. We would use ($\rho$, $\mu$, $L$) instead, and the final dimensionless function would become

$$C_F' = \frac{\rho F}{\mu^2} = \text{fcn(Re)} \qquad \text{Re} = \frac{\rho VL}{\mu} \tag{5.16}$$

In plotting these data, we would not be able to discern the effect of $\rho$ or $\mu$, since they appear in both dimensionless groups. The grouping $C_F'$ again would mean dimensionless force, and Re is now interpreted as either dimensionless velocity or size.[3] The plot would be quite different compared to Eq. (5.2), although it contains exactly the same information. The development of parameters such as $C_F'$ and Re from the initial variables is the subject of the pi theorem (Sec. 5.3).

## Some Peculiar Engineering Equations

The foundation of the dimensional-analysis method rests on two assumptions: (1) The proposed physical relation is dimensionally homogeneous, and (2) all the relevant variables have been included in the proposed relation.

If a relevant variable is missing, dimensional analysis will fail, giving either algebraic difficulties or, worse, yielding a dimensionless formulation which does not resolve the process. A typical case is Manning's open-channel formula, discussed in Example 1.4:

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{1}$$

Since $V$ is velocity, $R$ is a radius, and $n$ and $S$ are dimensionless, the formula is not dimensionally homogeneous. This should be a warning that (1) the formula changes if the *units* of $V$ and $R$ change and (2) if valid, it represents a very special case. Equation (1) in Example 1.4 (see above) predates the dimensional-analysis technique and is valid only for water in rough channels at moderate velocities and large radii in BG units.

Such dimensionally inhomogeneous formulas abound in the hydraulics literature. Another example is the Hazen-Williams formula [30] for volume flow of water through a straight smooth pipe

$$Q = 61.9 D^{2.63} \left(\frac{dp}{dx}\right)^{0.54} \tag{5.17}$$

where $D$ is diameter and $dp/dx$ is the pressure gradient. Some of these formulas arise because numbers have been inserted for fluid properties and other physical data into perfectly legitimate homogeneous formulas. We shall not give the units of Eq. (5.17) to avoid encouraging its use.

On the other hand, some formulas are "constructs" which cannot be made dimensionally homogeneous. The "variables" they relate cannot be analyzed by the dimensional-analysis technique. Most of these formulas are raw empiricisms convenient to a small group of specialists. Here are three examples:

$$B = \frac{25{,}000}{100 - R} \tag{5.18}$$

$$S = \frac{140}{130 + \text{API}} \tag{5.19}$$

$$0.0147 D_E - \frac{3.74}{D_E} = 0.26 t_R - \frac{172}{t_R} \tag{5.20}$$

Equation (5.18) relates the Brinell hardness $B$ of a metal to its Rockwell hardness $R$. Equation (5.19) relates the specific gravity $S$ of an oil to its density in degrees API.

[3] We were lucky to achieve a size effect because in this case L, a scaling parameter, did not appear in the drag coefficient.

Equation (5.20) relates the viscosity of a liquid in $D_E$, or degrees Engler, to its viscosity $t_R$ in Saybolt seconds. Such formulas have a certain usefulness when communicated between fellow specialists, but we cannot handle them here. Variables like Brinell hardness and Saybolt viscosity are not suited to an $MLT\Theta$ dimensional system.

## 5.3 The Pi Theorem

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups. The scheme given here was proposed in 1914 by Buckingham [29] and is now called the *Buckingham pi theorem*. The name *pi* comes from the mathematical notation $\Pi$, meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by $\Pi_1$, $\Pi_2$, $\Pi_3$, etc. The method allows the pis to be found in sequential order without resorting to free exponents.

The first part of the pi theorem explains what reduction in variables to expect:

> If a physical process satisfies the PDH and involves $n$ dimensional variables, it can be reduced to a relation between only $k$ dimensionless variables or $\Pi$'s. The reduction $j = n - k$ equals the maximum number of variables which do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Take the specific case of force on an immersed body: Eq. (5.1) contains five variables $F$, $L$, $U$, $\rho$, and $\mu$ described by three dimensions $\{MLT\}$. Thus $n = 5$ and $j \leq 3$. Therefore it is a good guess that we can reduce the problem to $k$ pis, with $k = n - j \geq 5 - 3 = 2$. And this is exactly what we obtained: two dimensionless variables $\Pi_1 = C_F$ and $\Pi_2 = \text{Re}$. On rare occasions it may take more pis than this minimum (see Example 5.5).

The second part of the theorem shows how to find the pis one at a time:

> Find the reduction $j$, then select $j$ scaling variables which do not form a pi among themselves.[4] Each desired pi group will be a power product of these $j$ variables plus one additional variable which is assigned any convenient nonzero exponent. Each pi group thus found is independent.

To be specific, suppose that the process involves five variables

$$v_1 = f(v_2, v_3, v_4, v_5)$$

Suppose that there are three dimensions $\{MLT\}$ and we search around and find that indeed $j = 3$. Then $k = 5 - 3 = 2$ and we expect, from the theorem, two and only two pi groups. Pick out three convenient variables which do *not* form a pi, and suppose these turn out to be $v_2$, $v_3$, and $v_4$. Then the two pi groups are formed by power products of these three plus one additional variable, either $v_1$ or $v_5$:

$$\Pi_1 = (v_2)^a (v_3)^b (v_4)^c v_1 = M^0 L^0 T^0 \qquad \Pi_2 = (v_2)^a (v_3)^b (v_4)^c v_5 = M^0 L^0 T^0$$

Here we have arbitrarily chosen $v_1$ and $v_5$, the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of $a$, $b$, and $c$ for each pi. And they are independent because only $\Pi_1$

[4] Make a clever choice here because all pis will contain these $j$ variables in various groupings.

**Table 5.1** Dimensions of Fluid-Mechanics Properties

| Quantity | Symbol | Dimensions $MLT\Theta$ | Dimensions $FLT\Theta$ |
|---|---|---|---|
| Length | $L$ | $L$ | $L$ |
| Area | $A$ | $L^2$ | $L^2$ |
| Volume | $\mathscr{V}$ | $L^3$ | $L^3$ |
| Velocity | $V$ | $LT^{-1}$ | $LT^{-1}$ |
| Acceleration | $dV/dt$ | $LT^{-2}$ | $LT^{-2}$ |
| Speed of sound | $a$ | $LT^{-1}$ | $LT^{-1}$ |
| Volume flow | $Q$ | $L^3T^{-1}$ | $L^3T^{-1}$ |
| Mass flow | $\dot{m}$ | $MT^{-1}$ | $FTL^{-1}$ |
| Pressure, stress | $p$, $\sigma$ | $ML^{-1}T^{-2}$ | $FL^{-2}$ |
| Strain rate | $\dot{\epsilon}$ | $T^{-1}$ | $T^{-1}$ |
| Angle | $\theta$ | None | None |
| Angular velocity | $\omega$ | $T^{-1}$ | $T^{-1}$ |
| Viscosity | $\mu$ | $ML^{-1}T^{-1}$ | $FTL^{-2}$ |
| Kinematic viscosity | $\nu$ | $L^2T^{-1}$ | $L^2T^{-1}$ |
| Surface tension | $\Upsilon$ | $MT^{-2}$ | $FL^{-1}$ |
| Force | $F$ | $MLT^{-2}$ | $F$ |
| Moment, torque | $M$ | $ML^2T^{-2}$ | $FL$ |
| Power | $P$ | $ML^2T^{-3}$ | $FLT^{-1}$ |
| Work, energy | $W$, $E$ | $ML^2T^{-2}$ | $FL$ |
| Density | $\rho$ | $ML^{-3}$ | $FT^2L^{-4}$ |
| Temperature | $T$ | $\Theta$ | $\Theta$ |
| Specific heat | $c_p$, $c_v$ | $L^2T^{-2}\Theta^{-1}$ | $L^2T^{-2}\Theta^{-1}$ |
| Specific weight | $\gamma$ | $ML^{-2}T^{-2}$ | $FL^{-3}$ |
| Thermal conductivity | $k$ | $MLT^{-3}\Theta^{-1}$ | $FT^{-1}\Theta^{-1}$ |
| Expansion coefficient | $\beta$ | $\Theta^{-1}$ | $\Theta^{-1}$ |

contains $\upsilon_1$ and only $\Pi_2$ contains $\upsilon_5$. It is a very neat system once you get used to the procedure. We shall illustrate it with several examples.

Typically, six steps are involved:

1. List and count the $n$ variables involved in the problem. If any important variables are missing, dimensional analysis will fail.
2. List the dimensions of each variable according to {MLTΘ} or {FLTΘ}. A list is given in Table 5.1.
3. Find $j$. Initially guess $j$ equal to the number of different dimensions present, and look for $j$ variables which do not form a pi product. If no luck, reduce $j$ by 1 and look again. With practice, you will find $j$ rapidly.
4. Select $j$ scaling parameters which do not form a pi product. Make sure they please you and have some generality if possible, because they will then appear in every one of your pi groups. Pick density or velocity or length. Do not pick surface tension, e.g., or you will form six different independent Weber-number parameters and thoroughly annoy your colleagues.
5. Add one additional variable to your $j$ repeating variables, and form a power product. Algebraically find the exponents which make the product dimensionless. Try to arrange for your output or *dependent* variables (force, pressure drop, torque, power) to appear in the numerator, and your plots will look better. Do

this sequentially, adding one new variable each time, and you will find all $n - j = k$ desired pi products.

6. Write the final dimensionless function, and check your work to make sure all pi groups are dimensionless.

## EXAMPLE 5.2

Repeat the development of Eq. (5.2) from Eq. (5.1), using the pi theorem.

### Solution

**Step 1** Write the function and count variables:

$$F = f(L, U, \rho, \mu) \quad \text{there are five variables } (n = 5)$$

**Step 2** List dimensions of each variable. From Table 5.1

| $F$ | $L$ | $U$ | $\rho$ | $\mu$ |
|---|---|---|---|---|
| $\{MLT^{-2}\}$ | $\{L\}$ | $\{LT^{-1}\}$ | $\{ML^{-3}\}$ | $\{ML^{-1}T^{-1}\}$ |

**Step 3** Find $j$. No variable contains the dimension $\Theta$, and so $j$ is less than or equal to 3 ($MLT$). We inspect the list and see that $L$, $U$, and $\rho$ cannot form a pi group because only $\rho$ contains mass and only $U$ contains time. Therefore $j$ does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

**Step 4** Select repeating $j$ variables. The group $L$, $U$, $\rho$ we found in step 3 will do fine.

**Step 5** Combine $L$, $U$, $\rho$ with one additional variable, in sequence, to find the two pi products.

First add force to find $\Pi_1$. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since $F$ is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

Equate exponents:

Length: $a + b - 3c + 1 = 0$

Mass: $c + 1 = 0$

Time: $-b \qquad -2 = 0$

We can solve explicitly for

$$a = -2 \qquad b = -2 \qquad c = -1$$

Therefore

$$\Pi_1 = L^{-2} U^{-2} \rho^{-1} F = \frac{F}{\rho U^2 L^2} = C_F \qquad \textit{Ans.}$$

This is exactly the right pi group as in Eq. (5.2). By varying the exponent on $F$, we could have found other equivalent groups such as $UL\rho^{1/2}/F^{1/2}$.

Finally, add viscosity to $L$, $U$, and $\rho$ to find $\Pi_2$. Select any power you like for viscosity. By hindsight and custom, we select the power $-1$ to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Equate exponents:

Length: $\quad a + b - 3c + 1 = 0$

Mass: $\quad c - 1 = 0$

Time: $\quad -b \quad + 1 = 0$

from which we find

$$a = b = c = 1$$

Therefore $\quad \Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \dfrac{\rho U L}{\mu} = \text{Re}$ *Ans.*

We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right)$$ *Ans.*

which is exactly Eq. (5.2).

## EXAMPLE 5.3

Reduce the falling-body relationship, Eq. (5.5), to a function of dimensionless variables. Why are there three different formulations?

### Solution

Write the function and count variables

$$S = f(t, S_0, V_0, g) \quad \text{five variables } (n = 5)$$

List the dimensions of each variable, from Table 5.1:

| $S$ | $t$ | $S_0$ | $V_0$ | $g$ |
|---|---|---|---|---|
| $\{L\}$ | $\{T\}$ | $\{L\}$ | $LT^{-1}\}$ | $\{LT^{-2}\}$ |

There are only two primary dimensions ($L$, $T$), so that $j \le 2$. By inspection we can easily find two variables which cannot be combined to form a pi, for example, $V_0$ and $g$. Then $j = 2$, and we expect $5 - 2 = 3$ pi products. Select $j$ variables among the parameters $S_0$, $V_0$, and $g$. Avoid $S$ and $t$ since they are the dependent variables, which should not be repeated in pi groups.

There are three different options for repeating variables among the group ($S_0$, $V_0$, $g$). Therefore we can obtain three different dimensionless formulations, just as we did informally with the falling-body equation in Sec. 5.2. Take each option in turn:

1. Choose $S_0$ and $V_0$ as repeating variables. Combine them in turn with ($S$, $t$, $g$):

$$\Pi_1 = S^1 S_0^a V_0^b \quad \Pi_2 = t^1 S_0^c V_0^d \quad \Pi_3 = g^1 S_0^e V_0^f$$

Set each power product equal to $L^0T^0$, and solve for the exponents ($a$, $b$, $c$, $d$, $e$, $f$). Please allow us to give the results here, and you may check the algebra as an exercise:

$$a = -1 \quad b = 0 \quad c = -1 \quad d = 1 \quad e = 1 \quad f = -2$$

$$\Pi_1 = S^* = \frac{S}{S_0} \qquad \Pi_2 = t^* = \frac{V_0 t}{S_0} \qquad \Pi_3 = \alpha = \frac{gS_0}{V_0^2} \qquad \textit{Ans.}$$

Thus, for option 1, we know that $S^* = \text{fcn}(t^*, \alpha)$. We have found, by dimensional analysis, the same variables as in Eq. (5.10). But here there is no *formula* for the functional relation — we might have to experiment with falling bodies to establish Fig. 5.1*a*.

2. Choose $V_0$ and $g$ as repeating variables. Combine them in turn with ($S$, $t$, $S_0$):

$$\Pi_1 = S^1 V_0^a g^b \quad \Pi_2 = t^1 V_0^c g^d \quad \Pi_3 = S_0^1 V_0^e g^f$$

Set each power product equal to $L^0T^0$, and solve for the exponents ($a$, $b$, $c$, $d$, $e$, $f$). Once more allow us to give the results here, and you may check the algebra as an exercise.

$$a = -2 \quad b = 1 \quad c = -1 \quad d = 1 \quad e = 1 \quad f = -2$$

$$\Pi_1 = S^{**} = \frac{Sg}{V_0^2} \quad \Pi_2 = t^{**} = \frac{tg}{V_0} \quad \Pi_3 = \alpha = \frac{gS_0}{V_0^2} \qquad \textit{Ans.}$$

Thus, for option 2, we now know that $S^{**} = \text{fcn}(t^{**}, \alpha)$. We have found, by dimensional analysis, the same groups as in Eq. (5.12). The data would plot as in Fig. 5.1*b*.

3. Finally choose $S_0$ and $g$ as repeating variables. Combine them in turn with ($S$, $t$, $V_0$):

$$\Pi_1 = S^1 S_0^a g^b \quad \Pi_2 = t^1 S_0^c g^d \quad \Pi_3 = V_0^1 S_0^e g^f$$

Set each power product equal to $L^0T^0$, and solve for the exponents ($a$, $b$, $c$, $d$, $e$, $f$). One more time allow us to give the results here, and you may check the algebra as an exercise:

$$a = -1 \quad b = 0 \quad c = -\frac{1}{2} \quad d = \frac{1}{2} \quad e = -\frac{1}{2} \quad f = -\frac{1}{2}$$

$$\Pi_1 = S^{***} = \frac{S}{S_0} \quad \Pi_2 = t^{***} = t\sqrt{\frac{g}{S_0}} \quad \Pi_3 = \beta = \frac{V_0}{\sqrt{gS_0}} \qquad \textit{Ans.}$$

Thus, for option 3, we now know that $S^{***} = \text{fcn}(t^{***}, \beta = 1/\sqrt{\alpha})$. We have found, by dimensional analysis, the same groups as in Eq. (5.14). The data would plot as in Fig. 5.1*c*.

Dimensional analysis here has yielded the same pi groups as the use of scaling parameters with Eq. (5.5). Three different formulations appeared, because we could choose three different pairs of repeating variables to complete the pi theorem.

## EXAMPLE 5.4

At low velocities (laminar flow), the volume flow $Q$ through a small-bore tube is a function only of the tube radius $R$, the fluid viscosity $\mu$, and the pressure drop per unit tube length $dp/dx$. Using the pi theorem, find an appropriate dimensionless relationship.

## Solution

Write the given relation and count variables:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right) \quad \text{four variables } (n = 4)$$

Make a list of the dimensions of these variables from Table 5.1:

| $Q$ | $R$ | $\mu$ | $dp/dx$ |
|---|---|---|---|
| $\{L^3T^{-1}\}$ | $\{L\}$ | $\{ML^{-1}T^{-1}\}$ | $\{ML^{-2}T^{-2}\}$ |

There are three primary dimensions ($M$, $L$, $T$), hence $j \le 3$. By trial and error we determine that $R$, $\mu$, and $dp/dx$ cannot be combined into a pi group. Then $j = 3$, and $n - j = 4 - 3 = 1$. There is only *one* pi group, which we find by combining $Q$ in a power product with the other three:

$$\Pi_1 = R^a \mu^b \left(\frac{dp}{dx}\right)^c Q^1 = (L)^a (ML^{-1}T^{-1})^b (ML^{-2}T^{-2})^c (L^3T^{-1})$$

$$= M^0L^0T^0$$

Equate exponents:

Mass: $\quad b + c = 0$

Length: $\quad a - b - 2c + 3 = 0$

Time: $\quad -b - 2c - 1 = 0$

Solving simultaneously, we obtain $a = -4$, $b = 1$, $c = -1$. Then

$$\Pi_1 = R^{-4} \mu^1 \left(\frac{dp}{dx}\right)^{-1} Q$$

or

$$\Pi_1 = \frac{Q\mu}{R^4(dp/dx)} = \text{const} \qquad \textit{Ans.}$$

Since there is only one pi group, it must equal a dimensionless constant. This is as far as dimensional analysis can take us. The laminar-flow theory of Sec. 6.4 shows that the value of the constant is $\pi/8$.

## EXAMPLE 5.5

Assume that the tip deflection $\delta$ of a cantilever beam is a function of the tip load $P$, beam length $L$, area moment of inertia $I$, and material modulus of elasticity $E$; that is, $\delta = f(P, L, I, E)$. Rewrite this function in dimensionless form, and comment on its complexity and the peculiar value of $j$.

## Solution

List the variables and their dimensions:

| $\delta$ | $P$ | $L$ | $I$ | $E$ |
|---|---|---|---|---|
| $\{L\}$ | $\{MLT^{-2}\}$ | $\{L\}$ | $\{L^4\}$ | $\{ML^{-1}T^{-2}\}$ |

There are five variables ($n = 5$) and three primary dimensions ($M$, $L$, $T$), hence $j \leq 3$. But try as we may, we *cannot* find any combination of three variables which does not form a pi group. This is because $\{M\}$ and $\{T\}$ occur only in $P$ and $E$ and only in the same form, $\{MT^{-2}\}$. Thus we have encountered a special case of $j = 2$, which is less than the number of dimensions ($M$, $L$, $T$). To gain more insight into this peculiarity, you should rework the problem, using the ($F$, $L$, $T$) system of dimensions.

With $j = 2$, we select $L$ and $E$ as two variables which cannot form a pi group and then add other variables to form the three desired pis:

$$\Pi_1 = L^a E^b I^1 = (L)^a (ML^{-1}T^{-2})^b (L^4) = M^0 L^0 T^0$$

from which, after equating exponents, we find that $a = -4$, $b = 0$, or $\Pi_1 = I/L^4$. Then

$$\Pi_2 = L^a E^b P^1 = (L)^a (ML^{-1}T^{-2})^b (MLT^{-2}) = M^0 L^0 T^0$$

from which we find $a = -2$, $b = -1$, or $\Pi_2 = P/(EL^2)$, and

$$\Pi_3 = L^a E^b \delta^1 = (L)^a (ML^{-1}T^{-2})^b (L) = M^0 L^0 T^0$$

from which $a = -1$, $b = 0$, or $\Pi_3 = \delta/L$. The proper dimensionless function is $\Pi_3 = f(\Pi_2, \Pi_1)$, or

$$\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right) \qquad \textit{Ans.} \quad (1)$$

This is a complex three-variable function, but dimensional analysis alone can take us no further.

We can "improve" Eq. (1) by taking advantage of some physical reasoning, as Langhaar points out [8, p. 91]. For small elastic deflections, $\delta$ is proportional to load $P$ and inversely proportional to moment of inertia $I$. Since $P$ and $I$ occur separately in Eq. (1), this means that $\Pi_3$ must be proportional to $\Pi_2$ and inversely proportional to $\Pi_1$. Thus, for these conditions,

$$\frac{\delta}{L} = (\text{const}) \frac{P}{EL^2} \frac{L^4}{I}$$

or

$$\delta = (\text{const}) \frac{PL^3}{EI} \qquad (2)$$

This could not be predicted by a pure dimensional analysis. Strength-of-materials theory predicts that the value of the constant is $\frac{1}{3}$.

## 5.4 Nondimensionalization of the Basic Equations

We could use the pi-theorem method of the previous section to analyze problem after problem after problem, finding the dimensionless parameters which govern in each case. Textbooks on dimensional analysis [for example, 7] do this. An alternate and very powerful technique is to attack the basic equations of flow from Chap. 4. Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, e.g., Reynolds number, in their proper form and proper position, giving clues to when they are negligible. The boundary conditions must also be nondimensionalized.

Let us briefly apply this technique to the incompressible-flow continuity and momentum equations with constant viscosity:

Continuity: $$\nabla \cdot \mathbf{V} = 0 \qquad (5.21a)$$

Momentum: $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{V} \tag{5.21$b$}$$

Typical boundary conditions for these two equations are

Fixed solid surface: $\mathbf{V} = 0$

Inlet or outlet: Known $\mathbf{V}$, $p$ (5.22)

Free surface, $z = \eta$: $w = \dfrac{d\eta}{dt} \quad p = p_a - \Upsilon(R_x{}^{-1} + R_y{}^{-1})$

We omit the energy equation (4.75) and assign its dimensionless form in the problems (Probs. 5.42 and 5.45).

Equations (5.21) and (5.22) contain the three basic dimensions $M$, $L$, and $T$. All variables $p$, $\mathbf{V}$, $x$, $y$, $z$, and $t$ can be nondimensionalized by using density and two reference constants which might be characteristic of the particular fluid flow:

Reference velocity $= U$ Reference length $= L$

For example, $U$ may be the inlet or upstream velocity and $L$ the diameter of a body immersed in the stream.

Now define all relevant dimensionless variables, denoting them by an asterisk:

$$\mathbf{V}^* = \frac{\mathbf{V}}{U}$$
$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad z^* = \frac{z}{L} \tag{5.23}$$
$$t^* = \frac{tU}{L} \quad p^* = \frac{p + \rho g z}{\rho U^2}$$

All these are fairly obvious except for $p^*$, where we have slyly introduced the gravity effect, assuming that $z$ is up. This is a hindsight idea suggested by Bernoulli's equation (3.77).

Since $\rho$, $U$, and $L$ are all constants, the derivatives in Eqs. (5.21) can all be handled in dimensionless form with dimensional coefficients. For example,

$$\frac{\partial u}{\partial x} = \frac{\partial(Uu^*)}{\partial(Lx^*)} = \frac{U}{L}\frac{\partial u^*}{\partial x^*}$$

Substitute the variables from Eqs. (5.23) into Eqs. (5.21) and (5.22) and divide through by the leading dimensional coefficient, in the same way as we handled Eq. (5.12). The resulting dimensionless equations of motion are:

Continuity: $$\boldsymbol{\nabla}^* \cdot \mathbf{V}^* = 0 \tag{5.24$a$}$$

Momentum: $$\frac{d\mathbf{V}^*}{dt^*} = -\boldsymbol{\nabla}^* p^* + \frac{\mu}{\rho U L}\nabla^{*2}(\mathbf{V}^*) \tag{5.24$b$}$$

The dimensionless boundary conditions are:

Fixed solid surface: $\mathbf{V}^* = 0$

Inlet or outlet: Known $\mathbf{V}^*$, $p^*$

Free surface, $z^* = \eta^*$: $$w^* = \frac{d\eta^*}{dt^*} \tag{5.25}$$

$$p^* = \frac{p_a}{\rho U^2} + \frac{gL}{U^2} z^* - \frac{\Upsilon}{\rho U^2 L} (R_x^{*-1} + R_y^{*-1})$$

These equations reveal a total of four dimensionless parameters, one in the momentum equation and three in the free-surface-pressure boundary condition.

## Dimensionless Parameters

In the continuity equation there are no parameters. The momentum equation contains one, generally accepted as the most important parameter in fluid mechanics:

$$\text{Reynolds number Re} = \frac{\rho U L}{\mu}$$

It is named after Osborne Reynolds (1842–1912), a British engineer who first proposed it in 1883 (Ref. 4 of Chap. 6). The Reynolds number is always important, with or without a free surface, and can be neglected only in flow regions away from high-velocity gradients, e.g., away from solid surfaces, jets, or wakes.

The no-slip and inlet-exit boundary conditions contain no parameters. The free-surface-pressure condition contains three:

$$\text{Euler number (pressure coefficient) Eu} = \frac{p_a}{\rho U^2}$$

This is named after Leonhard Euler (1707–1783) and is rarely important unless the pressure drops low enough to cause vapor formation (cavitation) in a liquid. The Euler number is often written in terms of pressure differences: Eu $= \Delta p/(\rho U^2)$. If $\Delta p$ involves vapor pressure $p_v$, it is called the *cavitation number* Ca $= (p_a - p_v)/(\rho U^2)$.

The second pressure parameter is much more important:

$$\text{Froude number Fr} = \frac{U^2}{gL}$$

It is named after William Froude (1810–1879), a British naval architect who, with his son Robert, developed the ship-model towing-tank concept and proposed similarity rules for free-surface flows (ship resistance, surface waves, open channels). The Froude number is the dominant effect in free-surface flows and is totally unimportant if there is no free surface. Chapter 10 investigates Froude number effects in detail.

The final free-surface parameter is

$$\text{Weber number We} = \frac{\rho U^2 L}{\Upsilon}$$

It is named after Moritz Weber (1871–1951) of the Polytechnic Institute of Berlin, who developed the laws of similitude in their modern form. It was Weber who named Re and Fr after Reynolds and Froude. The Weber number is important only if it is of order unity or less, which typically occurs when the surface curvature is comparable in size to the liquid depth, e.g., in droplets, capillary flows, ripple waves, and very small hydraulic models. If We is large, its effect may be neglected.

If there is no free surface, Fr, Eu, and We drop out entirely, except for the possibility of cavitation of a liquid at very small Eu. Thus, in low-speed viscous flows with no free surface, the Reynolds number is the only important dimensionless parameter.

## Compressibility Parameters

In high-speed flow of a gas there are significant changes in pressure, density, and temperature which must be related by an equation of state such as the perfect-gas law, Eq. (1.10). These thermodynamic changes introduce two additional dimensionless parameters mentioned briefly in earlier chapters:

$$\text{Mach number Ma} = \frac{U}{a} \qquad \text{Specific-heat ratio } k = \frac{c_p}{c_v}$$

The Mach number is named after Ernst Mach (1838–1916), an Austrian physicist. The effect of $k$ is only slight to moderate, but Ma exerts a strong effect on compressible-flow properties if it is greater than about 0.3. These effects are studied in Chap. 9.

## Oscillating Flows

If the flow pattern is oscillating, a seventh parameter enters through the inlet boundary condition. For example, suppose that the inlet stream is of the form

$$u = U \cos \omega t$$

Nondimensionalization of this relation results in

$$\frac{u}{U} = u^* = \cos\left(\frac{\omega L}{U} t^*\right)$$

The argument of the cosine contains the new parameter

$$\text{Strouhal number St} = \frac{\omega L}{U}$$

The dimensionless forces and moments, friction, and heat transfer, etc., of such an oscillating flow would be a function of both Reynolds and Strouhal numbers. This parameter is named after V. Strouhal, a German physicist who experimented in 1878 with wires singing in the wind.

Some flows which you might guess to be perfectly steady actually have an oscillatory pattern which is dependent on the Reynolds number. An example is the periodic vortex shedding behind a blunt body immersed in a steady stream of velocity $U$. Figure 5.2*a* shows an array of alternating vortices shed from a circular cylinder immersed in a steady crossflow. This regular, periodic shedding is called a *Kármán vortex street*, after T. von Kármán, who explained it theoretically in 1912. The shedding occurs in the range $10^2 < \text{Re} < 10^7$, with an average Strouhal number $\omega d/(2\pi U) \approx 0.21$. Figure 5.2*b* shows measured shedding frequencies.

Resonance can occur if a vortex shedding frequency is near a body's structural-vibration frequency. Electric transmission wires sing in the wind, undersea mooring lines gallop at certain current speeds, and slender structures flutter at critical wind or vehicle speeds. A striking example is the disastrous failure of the Tacoma Narrows suspension bridge in 1940, when wind-excited vortex shedding caused resonance with the natural torsional oscillations of the bridge.

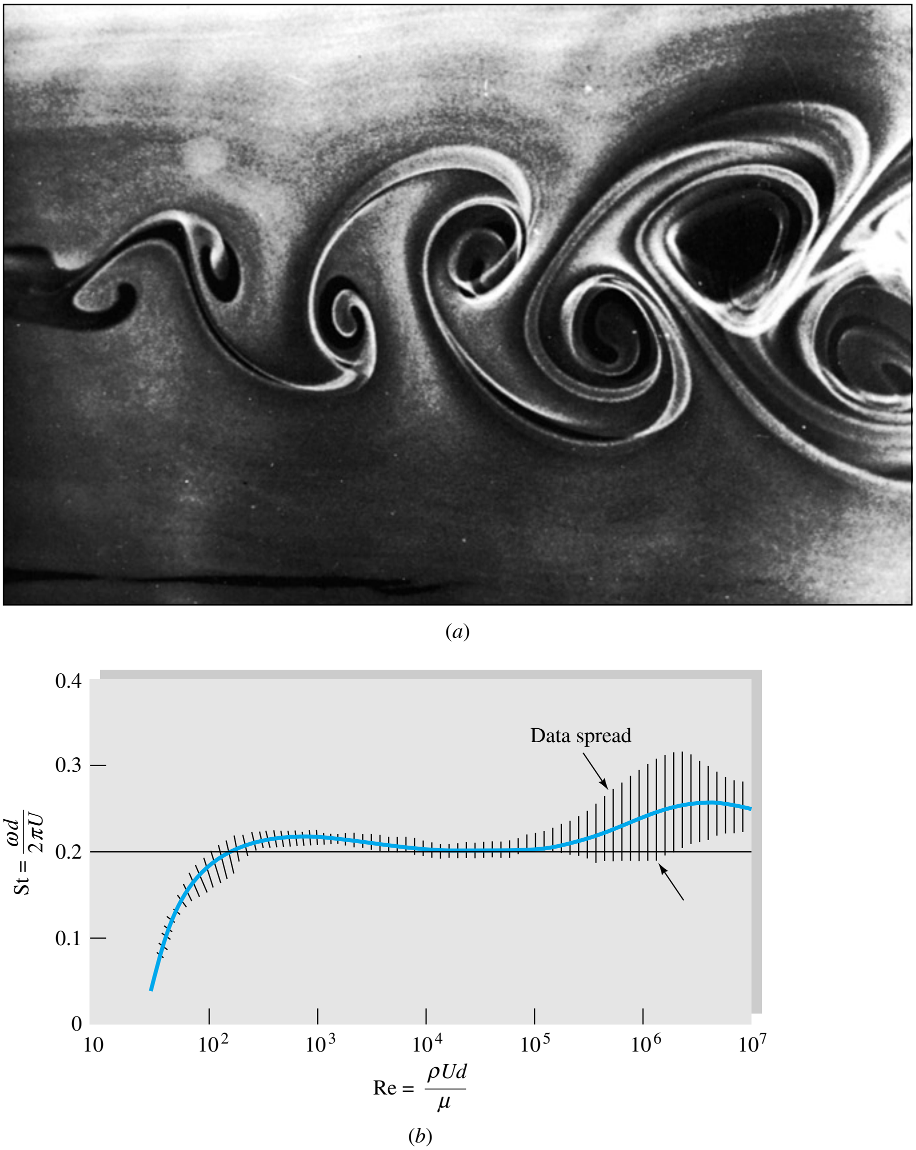



**Fig. 5.2** Vortex shedding from a circular cylinder: (*a*) vortex street behind a circular cylinder *(from Ref. 33, courtesy of U.S. Naval Research Laboratory);* (*b*) experimental shedding frequencies *(data from Refs. 31 and 32).*

## Other Dimensionless Parameters

We have discussed seven important parameters in fluid mechanics, and there are others. Four additional parameters arise from nondimensionalization of the energy equation (4.75) and its boundary conditions. These four (Prandtl number, Eckert number, Grashof number, and wall-temperature ratio) are listed in Table 5.2 just in case you fail to solve Prob. 5.42. Another important and rather sneaky parameter is the wall-roughness ratio $\epsilon/L$ (in Table 5.2).[5] Slight changes in surface roughness have a strik-

[5] Roughness is easy to overlook because it is a slight geometric effect which does not appear in the equations of motion.

**Table 5.2** Dimensionless Groups in Fluid Mechanics

| Parameter | Definition | Qualitative ratio of effects | Importance |
|---|---|---|---|
| Reynolds number | $\text{Re} = \frac{\rho U L}{\mu}$ | $\frac{\text{Inertia}}{\text{Viscosity}}$ | Always |
| Mach number | $\text{Ma} = \frac{U}{a}$ | $\frac{\text{Flow speed}}{\text{Sound speed}}$ | Compressible flow |
| Froude number | $\text{Fr} = \frac{U^2}{gL}$ | $\frac{\text{Inertia}}{\text{Gravity}}$ | Free-surface flow |
| Weber number | $\text{We} = \frac{\rho U^2 L}{\Upsilon}$ | $\frac{\text{Inertia}}{\text{Surface tension}}$ | Free-surface flow |
| Cavitation number (Euler number) | $\text{Ca} = \frac{p - p_v}{\rho U^2}$ | $\frac{\text{Pressure}}{\text{Inertia}}$ | Cavitation |
| Prandtl number | $\text{Pr} = \frac{\mu c_p}{k}$ | $\frac{\text{Dissipation}}{\text{Conduction}}$ | Heat convection |
| Eckert number | $\text{Ec} = \frac{U^2}{c_p T_0}$ | $\frac{\text{Kinetic energy}}{\text{Enthalpy}}$ | Dissipation |
| Specific-heat ratio | $k = \frac{c_p}{c_v}$ | $\frac{\text{Enthalpy}}{\text{Internal energy}}$ | Compressible flow |
| Strouhal number | $\text{St} = \frac{\omega L}{U}$ | $\frac{\text{Oscillation}}{\text{Mean speed}}$ | Oscillating flow |
| Roughness ratio | $\frac{\epsilon}{L}$ | $\frac{\text{Wall roughness}}{\text{Body length}}$ | Turbulent, rough walls |
| Grashof number | $\text{Gr} = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$ | $\frac{\text{Buoyancy}}{\text{Viscosity}}$ | Natural convection |
| Temperature ratio | $\frac{T_w}{T_0}$ | $\frac{\text{Wall temperature}}{\text{Stream temperature}}$ | Heat transfer |
| Pressure coefficient | $C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$ | $\frac{\text{Static pressure}}{\text{Dynamic pressure}}$ | Aerodynamics, hydrodynamics |
| Lift coefficient | $C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$ | $\frac{\text{Lift force}}{\text{Dynamic force}}$ | Aerodynamics, hydrodynamics |
| Drag coefficient | $C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$ | $\frac{\text{Drag force}}{\text{Dynamic force}}$ | Aerodynamics, hydrodynamics |

ing effect in the turbulent-flow or high-Reynolds-number range, as we shall see in Chap. 6 and in Fig. 5.3.

This book is primarily concerned with Reynolds-, Mach-, and Froude-number effects, which dominate most flows. Note that we discovered all these parameters (except $\epsilon/L$) simply by nondimensionalizing the basic equations without actually solving them.

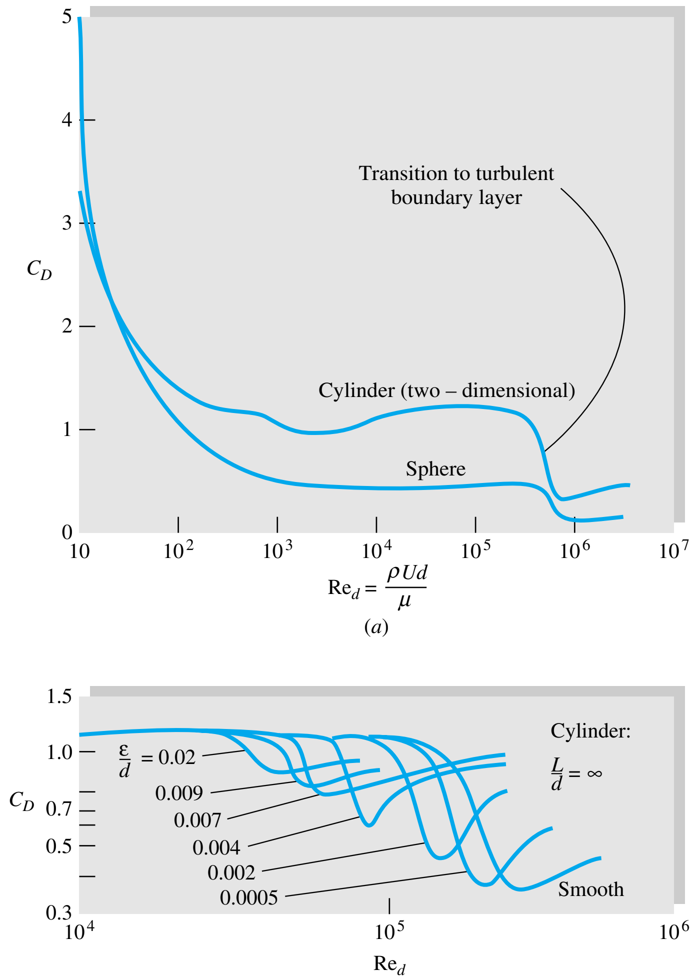
If the reader is not satiated with the 15 parameters given in Table 5.2, Ref. 34 contains a list of over 300 dimensionless parameters in use in engineering. See also Ref. 35.

## A Successful Application

Dimensional analysis is fun, but does it work? Yes; if all important variables are included in the proposed function, the dimensionless function found by dimensional analysis will collapse all the data onto a single curve or set of curves.

An example of the success of dimensional analysis is given in Fig. 5.3 for the measured drag on smooth cylinders and spheres. The flow is normal to the axis of the cylinder, which is extremely long, $L/d \to \infty$. The data are from many sources, for both liquids and gases, and include bodies from several meters in diameter down to fine wires and balls less than 1 mm in size. Both curves in Fig. 5.3*a* are entirely experimental; the analysis of immersed body drag is one of the weakest areas of modern fluid-mechanics theory. Except for some isolated digital-computer calculations, there is no theory for cylinder and sphere drag except *creeping flow,* $\text{Re} < 1$.

The Reynolds number of both bodies is based upon diameter, hence the notation $\text{Re}_d$. But the drag coefficients are defined differently:




Cylinder length effect

$(10^4 < \text{Re} < 10^5)$

| $L/d$ | $C_D$ |
|---|---|
| $\infty$ | 1.20 |
| 40 | 0.98 |
| 20 | 0.91 |
| 10 | 0.82 |
| 5 | 0.74 |
| 3 | 0.72 |
| 2 | 0.68 |
| 1 | 0.64 |

**Fig. 5.3** The proof of practical dimensional analysis: drag coefficients of a cylinder and sphere: (*a*) drag coefficient of a smooth cylinder and sphere (data from many sources); (*b*) increased roughness causes earlier transition to a turbulent boundary layer.

$$C_D = \begin{cases} \dfrac{\text{drag}}{\frac{1}{2}\rho U^2 L d} & \text{cylinder} \\ \dfrac{\text{drag}}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2} & \text{sphere} \end{cases} \tag{5.26}$$

They both have a factor $\frac{1}{2}$ as a traditional tribute to Bernoulli and Euler, and both are based on the projected area, i.e., the area one sees when looking toward the body from upstream. The usual definition of $C_D$ is thus

$$C_D = \frac{\text{drag}}{\frac{1}{2}\rho U^2(\text{projected area})} \tag{5.27}$$

However, one should carefully check the definitions of $C_D$, Re, etc., before using data in the literature. Airfoils, e.g., use the planform area.

Figure 5.3*a* is for long, smooth cylinders. If wall roughness and cylinder length are included as variables, we obtain from dimensional analysis a complex three-parameter function

$$C_D = f\left(\text{Re}_d, \frac{\epsilon}{d}, \frac{L}{d}\right) \tag{5.28}$$

To describe this function completely would require 1000 or more experiments. Therefore it is customary to explore the length and roughness effects separately to establish trends.

The table with Fig. 5.3*a* shows the length effect with zero wall roughness. As length decreases, the drag decreases by up to 50 percent. Physically, the pressure is "relieved" at the ends as the flow is allowed to skirt around the tips instead of deflecting over and under the body.

Figure 5.3*b* shows the effect of wall roughness for an infinitely long cylinder. The sharp drop in drag occurs at lower $\text{Re}_d$ as roughness causes an earlier transition to a turbulent boundary layer on the surface of the body. Roughness has the same effect on sphere drag, a fact which is exploited in sports by deliberate dimpling of golf balls to give them less drag at their flight $\text{Re}_d \approx 10^5$.

Figure 5.3 is a typical experimental study of a fluid-mechanics problem, aided by dimensional analysis. As time and money and demand allow, the complete three-parameter relation (5.28) could be filled out by further experiments.

## EXAMPLE 5.6

The capillary rise $h$ of a liquid in a tube varies with tube diameter $d$, gravity $g$, fluid density $\rho$, surface tension $\Upsilon$, and the contact angle $\theta$. (*a*) Find a dimensionless statement of this relation. (*b*) If $h = 3$ cm in a given experiment, what will $h$ be in a similar case if the diameter and surface tension are half as much, the density is twice as much, and the contact angle is the same?

### Solution

**Part (a) Step 1** Write down the function and count variables:

$$h = f(d, g, \rho, \Upsilon, \theta) \qquad n = 6 \text{ variables}$$

**Step 2** List the dimensions {*FLT*} from Table 5.2:

| $h$ | $d$ | $g$ | $\rho$ | $\Upsilon$ | $\theta$ |
|---|---|---|---|---|---|
| $\{L\}$ | $\{L\}$ | $\{LT^{-2}\}$ | $\{FT^2L^{-4}\}$ | $\{FL^{-1}\}$ | none |

**Step 3** Find $j$. Several groups of three form no pi: $\Upsilon$, $\rho$, and $g$ or $\rho$, $g$, and $d$. Therefore $j = 3$, and we expect $n - j = 6 - 3 = 3$ dimensionless groups. One of these is obviously $\theta$, which is already dimensionless:

$$\Pi_3 = \theta \qquad \textit{Ans. (a)}$$

If we had carelessly chosen to search for it by using steps 4 and 5, we would still find $\Pi_3 = \theta$.

**Step 4** Select $j$ repeating variables which do not form a pi group: $\rho$, $g$, $d$.

**Step 5** Add one additional variable in sequence to form the pis:

Add $h$: $$\Pi_1 = \rho^a g^b d^c h = (FT^2L^{-4})^a(LT^{-2})^b(L)^c(L) = F^0L^0T^0$$

Solve for

$$a = b = 0 \qquad c = -1$$

Therefore $$\Pi_1 = \rho^0 g^0 d^{-1} h = \frac{h}{d} \qquad \textit{Ans. (a)}$$

Finally add $\Upsilon$, again selecting its exponent to be 1:

$$\Pi_2 = \rho^a g^b d^c \Upsilon = (FT^2L^{-4})^a(LT^{-2})^b(L)^c(FL^{-1}) = F^0L^0T^0$$

Solve for

$$a = b = -1 \quad c = -2$$

Therefore $$\Pi_2 = \rho^{-1} g^{-1} d^{-2} \Upsilon = \frac{\Upsilon}{\rho g d^2} \qquad \textit{Ans. (a)}$$

**Step 6** The complete dimensionless relation for this problem is thus

$$\frac{h}{d} = F\left(\frac{\Upsilon}{\rho g d^2}, \theta\right) \qquad \textit{Ans. (a)} \quad (1)$$

This is as far as dimensional analysis goes. Theory, however, establishes that $h$ is proportional to $\Upsilon$. Since $\Upsilon$ occurs only in the second parameter, we can slip it outside

$$\left(\frac{h}{d}\right)_{\text{actual}} = \frac{\Upsilon}{\rho g d^2} F_1(\theta) \qquad \text{or} \qquad \frac{h\rho g d}{\Upsilon} = F_1(\theta)$$

Example 1.9 showed theoretically that $F_1(\theta) = 4 \cos \theta$.

**Part (b)** We are given $h_1$ for certain conditions $d_1$, $\Upsilon_1$, $\rho_1$, and $\theta_1$. If $h_1 = 3$ cm, what is $h_2$ for $d_2 = \frac{1}{2}d_1$, $\Upsilon_2 = \frac{1}{2}\Upsilon_1$, $\rho_2 = 2\rho_1$, and $\theta_2 = \theta_1$? We know the functional relation, Eq. (1), must still hold at condition 2

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_2}{\rho_2 g d_2^2}, \theta_2\right)$$

But

$$\frac{\Upsilon_2}{\rho_2 g d_2^2} = \frac{\frac{1}{2}\Upsilon_1}{2\rho_1 g(\frac{1}{2}d_1)^2} = \frac{\Upsilon_1}{\rho_1 g d_1^2}$$

Therefore, functionally,

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_1}{\rho_1 g d_1^2}, \theta_1\right) = \frac{h_1}{d_1}$$

We are given a condition 2 which is exactly similar to condition 1, and therefore a scaling law holds

$$h_2 = h_1 \frac{d_2}{d_1} = (3 \text{ cm}) \frac{\frac{1}{2}d_1}{d_1} = 1.5 \text{ cm} \qquad \textit{Ans. (b)}$$

If the pi groups had not been exactly the same for both conditions, we would have had to know more about the functional relation $F$ to calculate $h_2$.

## 5.5 Modeling and Its Pitfalls

So far we have learned about dimensional homogeneity and the pi-theorem method, using power products, for converting a homogeneous physical relation to dimensionless form. This is straightforward mathematically, but there are certain engineering difficulties which need to be discussed.

First, we have more or less taken for granted that the variables which affect the process can be listed and analyzed. Actually, selection of the important variables requires considerable judgment and experience. The engineer must decide, e.g., whether viscosity can be neglected. Are there significant temperature effects? Is surface tension important? What about wall roughness? Each pi group which is retained increases the expense and effort required. Judgment in selecting variables will come through practice and maturity; this book should provide some of the necessary experience.

Once the variables are selected and the dimensional analysis is performed, the experimenter seeks to achieve *similarity* between the model tested and the prototype to be designed. With sufficient testing, the model data will reveal the desired dimensionless function between variables

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots \Pi_k) \qquad (5.29)$$

With Eq. (5.29) available in chart, graphical, or analytical form, we are in a position to ensure complete similarity between model and prototype. A formal statement would be as follows:

> Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype.

This follows mathematically from Eq. (5.29). If $\Pi_{2m} = \Pi_{2p}$, $\Pi_{3m} = \Pi_{3p}$, etc., Eq. (5.29) guarantees that the desired output $\Pi_{1m}$ will equal $\Pi_{1p}$. But this is easier said than done, as we now discuss.

Instead of complete similarity, the engineering literature speaks of particular types of similarity, the most common being geometric, kinematic, dynamic, and thermal. Let us consider each separately.

### Geometric Similarity

Geometric similarity concerns the length dimension $\{L\}$ and must be ensured before any sensible model testing can proceed. A formal definition is as follows:

A model and prototype are *geometrically similar* if and only if all body dimensions in all three coordinates have the same linear-scale ratio.

Note that *all* length scales must be the same. It is as if you took a photograph of the prototype and reduced it or enlarged it until it fitted the size of the model. If the model is to be made one-tenth the prototype size, its length, width, and height must each be one-tenth as large. Not only that, but also its entire shape must be one-tenth as large, and technically we speak of *homologous* points, which are points that have the same relative location. For example, the nose of the prototype is homologous to the nose of the model. The left wingtip of the prototype is homologous to the left wingtip of the model. Then geometric similarity requires that all homologous points be related by the same linear-scale ratio. This applies to the fluid geometry as well as the model geometry.

All angles are preserved in geometric similarity. All flow directions are preserved. The orientations of model and prototype with respect to the surroundings must be identical.

Figure 5.4 illustrates a prototype wing and a one-tenth-scale model. The model lengths are all one-tenth as large, but its angle of attack with respect to the free stream is the same: 10° not 1°. All physical details on the model must be scaled, and some are rather subtle and sometimes overlooked:

1. The model nose radius must be one-tenth as large.
2. The model surface roughness must be one-tenth as large.
3. If the prototype has a 5-mm boundary-layer trip wire 1.5 m from the leading edge, the model should have a 0.5-mm trip wire 0.15 m from its leading edge.
4. If the prototype is constructed with protruding fasteners, the model should have homologous protruding fasteners one-tenth as large.

And so on. Any departure from these details is a violation of geometric similarity and must be justified by experimental comparison to show that the prototype behavior was not significantly affected by the discrepancy.

Models which appear similar in shape but which clearly violate geometric similarity should not be compared except at your own risk. Figure 5.5 illustrates this point.

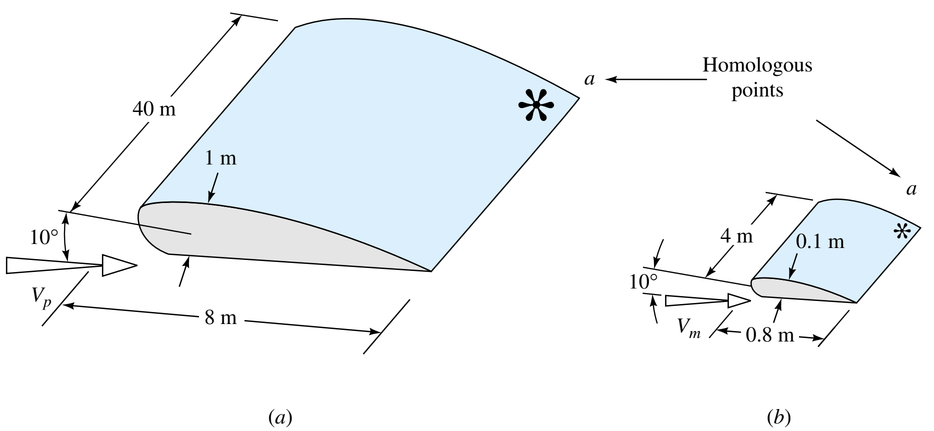



**Fig. 5.4** Geometric similarity in model testing: (*a*) prototype; (*b*) one-tenth-scale model.

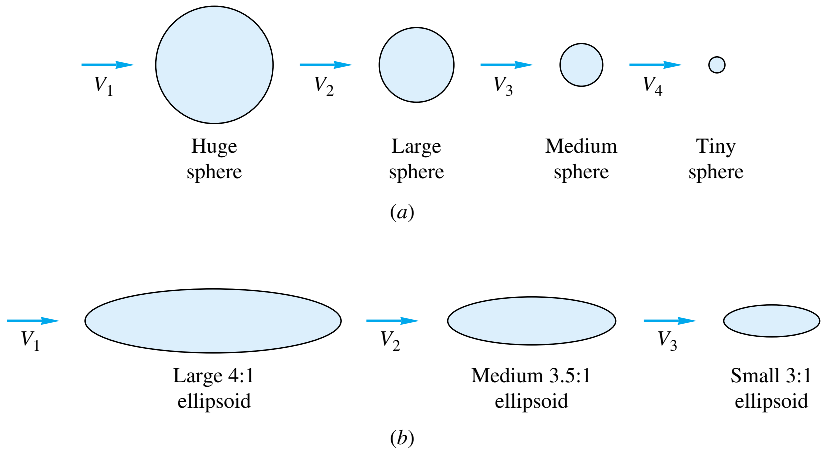



**Fig. 5.5** Geometric similarity and dissimilarity of flows: (*a*) similar; (*b*) dissimilar.

The spheres in Fig. 5.5*a* are all geometrically similar and can be tested with a high expectation of success if the Reynolds number or Froude number, etc., is matched. But the ellipsoids in Fig. 5.5*b* merely *look* similar. They actually have different linear-scale ratios and therefore cannot be compared in a rational manner, even though they may have identical Reynolds and Froude numbers, etc. The data will not be the same for these ellipsoids, and any attempt to "compare" them is a matter of rough engineering judgment.

## Kinematic Similarity

Kinematic similarity requires that the model and prototype have the same length-scale ratio and the same time-scale ratio. The result is that the velocity-scale ratio will be the same for both. As Langhaar [8] states it:

> The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times.

Length-scale equivalence simply implies geometric similarity, but time-scale equivalence may require additional dynamic considerations such as equivalence of the Reynolds and Mach numbers.

One special case is incompressible frictionless flow with no free surface, as sketched in Fig. 5.6*a*. These perfect-fluid flows are kinematically similar with independent length and time scales, and no additional parameters are necessary (see Chap. 8 for further details).

Frictionless flows with a free surface, as in Fig. 5.6*b*, are kinematically similar if their Froude numbers are equal

$$\mathrm{Fr}_m = \frac{V_m^2}{gL_m} = \frac{V_p^2}{gL_p} = \mathrm{Fr}_p \tag{5.30}$$

Note that the Froude number contains only length and time dimensions and hence is a purely kinematic parameter which fixes the relation between length and time. From Eq. (5.30), if the length scale is

$$L_m = \alpha L_p \tag{5.31}$$

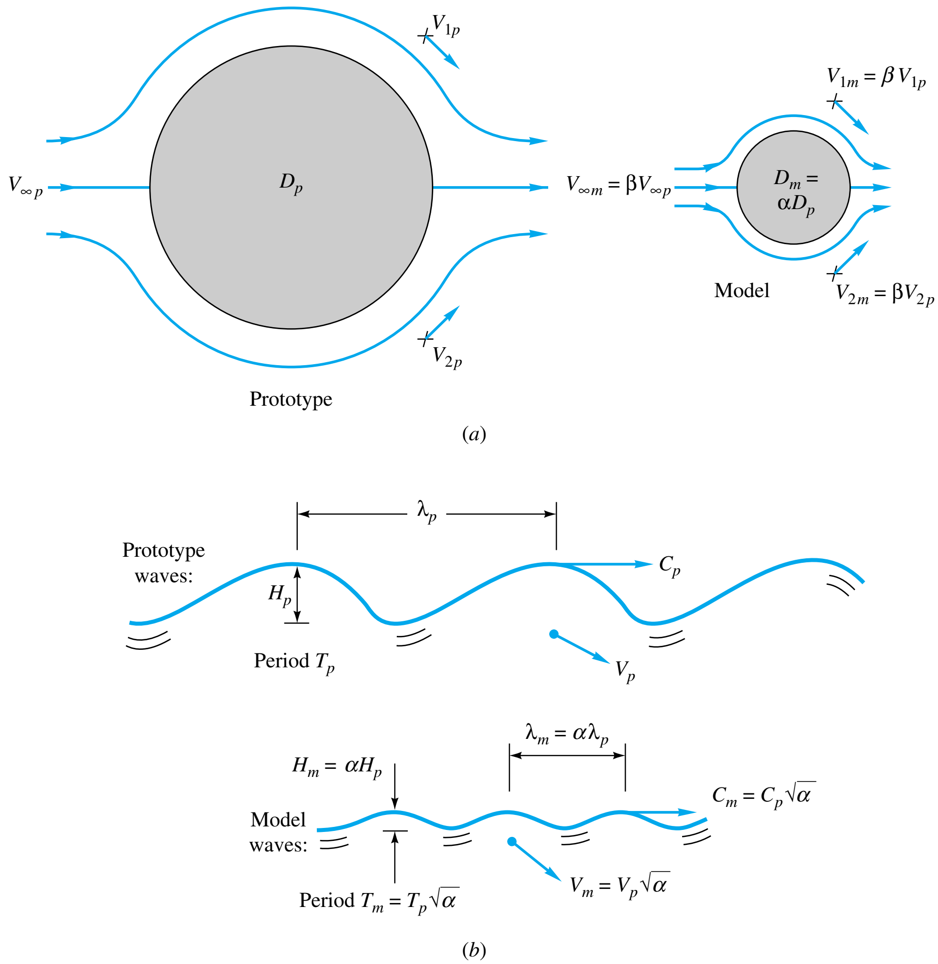



**Fig. 5.6** Frictionless low-speed flows are kinematically similar: (*a*) Flows with no free surface are kinematically similar with independent length- and time-scale ratios; (*b*) free-surface flows are kinematically similar with length and time scales related by the Froude number.

where $\alpha$ is a dimensionless ratio, the velocity scale is

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = \sqrt{\alpha} \tag{5.32}$$

and the time scale is

$$\frac{T_m}{T_p} = \frac{L_m/V_m}{L_p/V_p} = \sqrt{\alpha} \tag{5.33}$$

These Froude-scaling kinematic relations are illustrated in Fig. 5.6*b* for wave-motion modeling. If the waves are related by the length scale $\alpha$, then the wave period, propagation speed, and particle velocities are related by $\sqrt{\alpha}$.

If viscosity, surface tension, or compressibility is important, kinematic similarity is dependent upon the achievement of dynamic similarity.

## Dynamic Similarity

Dynamic similarity exists when the model and the prototype have the same length-scale ratio, time-scale ratio, and force-scale (or mass-scale) ratio. Again geometric similarity is a first requirement; without it, proceed no further. Then dynamic similarity exists, simultaneous with kinematic similarity, if the model and prototype force and pressure coefficients are identical. This is ensured if:

1. For compressible flow, the model and prototype Reynolds number and Mach number and specific-heat ratio are correspondingly equal.
2. For incompressible flow
   *a.* With no free surface: model and prototype Reynolds numbers are equal.
   *b.* With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.

Mathematically, Newton's law for any fluid particle requires that the sum of the pressure force, gravity force, and friction force equal the acceleration term, or inertia force,

$$\mathbf{F}_p + \mathbf{F}_g + \mathbf{F}_f = \mathbf{F}_i$$

The dynamic-similarity laws listed above ensure that each of these forces will be in the same ratio and have equivalent directions between model and prototype. Figure 5.7 shows an example for flow through a sluice gate. The force polygons at homologous points have exactly the same shape if the Reynolds and Froude numbers are equal (neglecting surface tension and cavitation, of course). Kinematic similarity is also ensured by these model laws.

## Discrepancies in Water and Air Testing

The perfect dynamic similarity shown in Fig. 5.7 is more of a dream than a reality because true equivalence of Reynolds and Froude numbers can be achieved only by dramatic changes in fluid properties, whereas in fact most model testing is simply done with water or air, the cheapest fluids available.

First consider hydraulic model testing with a free surface. Dynamic similarity requires equivalent Froude numbers, Eq. (5.30), *and* equivalent Reynolds numbers

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \tag{5.34}$$

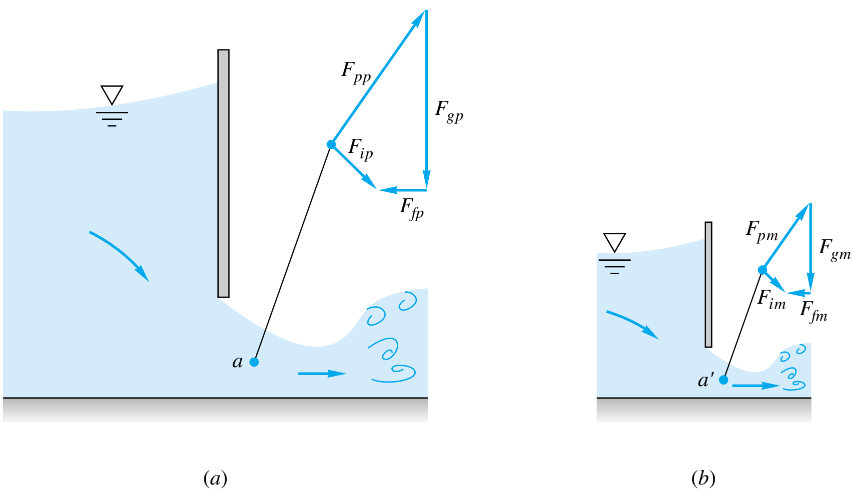



**Fig. 5.7** Dynamic similarity in sluice-gate flow. Model and prototype yield identical homologous force polygons if the Reynolds and Froude numbers are the same corresponding values: (*a*) prototype; (*b*) model.

But both velocity and length are constrained by the Froude number, Eqs. (5.31) and (5.32). Therefore, for a given length-scale ratio $\alpha$, Eq. (5.34) is true only if

$$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\frac{V_m}{V_p} = \alpha\sqrt{\alpha} = \alpha^{3/2} \tag{5.35}$$

For example, for a one-tenth-scale model, $\alpha = 0.1$ and $\alpha^{3/2} = 0.032$. Since $\nu_p$ is undoubtedly water, we need a fluid with only 0.032 times the kinematic viscosity of water to achieve dynamic similarity. Referring to Table 1.4, we see that this is impossible: Even mercury has only one-ninth the kinematic viscosity of water, and a mercury hydraulic model would be expensive and bad for your health. In practice, water is used for both the model and the prototype, and the Reynolds-number similarity (5.34) is unavoidably violated. The Froude number is held constant since it is the dominant parameter in free-surface flows. Typically the Reynolds number of the model flow is too small by a factor of 10 to 1000. As shown in Fig. 5.8, the low-Reynolds-number model data are used to estimate by extrapolation the desired high-Reynolds-number prototype data. As the figure indicates, there is obviously considerable uncertainty in using such an extrapolation, but there is no other practical alternative in hydraulic model testing.

Second, consider aerodynamic model testing in air with no free surface. The important parameters are the Reynolds number and the Mach number. Equation (5.34) should be satisfied, plus the compressibility criterion

$$\frac{V_m}{a_m} = \frac{V_p}{a_p} \tag{5.36}$$

Elimination of $V_m/V_p$ between (5.34) and (5.36) gives

$$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\frac{a_m}{a_p} \tag{5.37}$$

Since the prototype is no doubt an air operation, we need a wind-tunnel fluid of low viscosity and high speed of sound. Hydrogen is the only practical example, but clearly it is too expensive and dangerous. Therefore wind tunnels normally operate with air as the working fluid. Cooling and pressurizing the air will bring Eq. (5.37) into better

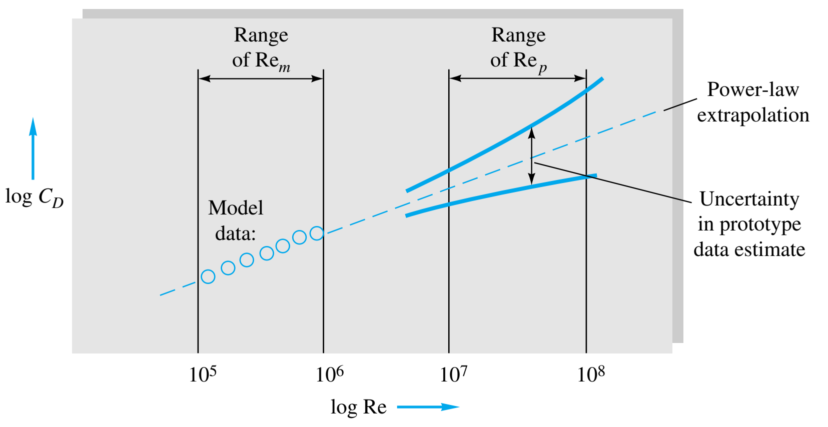



**Fig. 5.8** Reynolds-number extrapolation, or scaling, of hydraulic data with equal Froude numbers.


**Fig. 5.9** Hydraulic model of a barrier-beach inlet at Little River, South Carolina. Such models of necessity violate geometric similarity and do not model the Reynolds number of the prototype inlet. *(Courtesy of U.S. Army Engineer Waterways Experiment Station).*

agreement but not enough to satisfy a length-scale reduction of, say, one-tenth. Therefore Reynolds-number scaling is also commonly violated in aerodynamic testing, and an extrapolation like that in Fig. 5.8 is required here also.

Finally, a serious discrepancy of another type occurs in hydraulic models of natural flow systems such as rivers, harbors, estuaries, and embayments. Such flows have large horizontal dimensions and small relative vertical dimensions. If we were to scale an estuary model by a uniform linear length ratio of, say, 1 : 1000, the resulting model would be only a few millimeters deep and dominated by entirely spurious surface-tension or Weber-number effects. Therefore such hydraulic models commonly violate *geometric* similarity by "distorting" the vertical scale by a factor of 10 or more. Figure 5.9 shows a hydraulic model of a barrier-beach inlet in South Carolina. The horizontal scale reduction is 1 : 300, but the vertical scale is only 1 : 60. Since a deeper channel flows more efficiently, the model channel bottom is deliberately roughened more than the natural channel to correct for the geometric discrepancy. Thus the friction effect of the discrepancy can be corrected, but its effect on, say, dispersion of heat and mass is less well known.

## EXAMPLE 5.7

The pressure drop due to friction for flow in a long smooth pipe is a function of average flow velocity, density, viscosity, and pipe length and diameter: $\Delta p = \text{fcn}(V, \rho, \mu, L, D)$. We wish to know how $\Delta p$ varies with $V$. (*a*) Use the pi theorem to rewrite this function in dimensionless form. (*b*) Then plot this function, using the following data for three pipes and three fluids:

| $D$, cm | $L$, m | $Q$, m³/h | $\Delta p$, Pa | $\rho$, kg/m³ | $\mu$, kg/(m · s) | $V$, m/s* |
|---|---|---|---|---|---|---|
| 1.0 | 5.0 | 0.3 | 4,680 | 680† | 2.92 E-4† | 1.06 |
| 1.0 | 7.0 | 0.6 | 22,300 | 680† | 2.92 E-4† | 2.12 |
| 1.0 | 9.0 | 1.0 | 70,800 | 680† | 2.92 E-4† | 3.54 |
| 2.0 | 4.0 | 1.0 | 2,080 | 998‡ | 0.0010‡ | 0.88 |
| 2.0 | 6.0 | 2.0 | 10,500 | 998‡ | 0.0010‡ | 1.77 |
| 2.0 | 8.0 | 3.1 | 30,400 | 998‡ | 0.0010‡ | 2.74 |
| 3.0 | 3.0 | 0.5 | 540 | 13,550§ | 1.56 E-3§ | 0.20 |
| 3.0 | 4.0 | 1.0 | 2,480 | 13,550§ | 1.56 E-3§ | 0.39 |
| 3.0 | 5.0 | 1.7 | 9,600 | 13,550§ | 1.56 E-3§ | 0.67 |

*$V = Q/A$, $A = \pi D^2/4$.

†Gasoline.

‡Water.

§Mercury.

(*c*) Suppose it is further known that $\Delta p$ is proportional to $L$ (which is quite true for long pipes with well-rounded entrances). Use this information to simplify and improve the pi-theorem formulation. Plot the dimensionless data in this improved manner and comment upon the results.

## Solution

There are six variables with three primary dimensions involved $\{MLT\}$. Therefore we expect that $j = 6 - 3 = 3$ pi groups. We are correct, for we can find three variables which do not form a pi product, for example, $(\rho, V, L)$. Carefully select three ($j$) repeating variables, but not including $\Delta p$ or $V$, which we plan to plot versus each other. We select $(\rho, \mu, D)$, and the pi theorem guarantees that three independent power-product groups will occur:

$$\Pi_1 = \rho^a \mu^b D^c \,\Delta p \qquad \Pi_2 = \rho^d \mu^e D^f V \qquad \Pi_3 = \rho^g \mu^h D^i L$$

or

$$\Pi_1 = \frac{\rho D^2\,\Delta p}{\mu^2} \qquad \Pi_2 = \frac{\rho V D}{\mu} \qquad \Pi_3 = \frac{L}{D}$$

We have omitted the algebra of finding $(a, b, c, d, e, f, g, h, i)$ by setting all exponents to zero $M^0$, $L^0$, $T^0$. Therefore we wish to plot the dimensionless relation

$$\frac{\rho D^2\,\Delta p}{\mu^2} = \text{fcn}\left(\frac{\rho V D}{\mu}, \frac{L}{D}\right) \qquad \textit{Ans. (a)}$$

We plot $\Pi_1$ versus $\Pi_2$ with $\Pi_3$ as a parameter. There will be nine data points. For example, the first row in the data above yields

$$\frac{\rho D^2\,\Delta p}{\mu^2} = \frac{(680)(0.01)^2(4680)}{(2.92\text{ E-4})^2} = 3.73\text{ E9}$$

$$\frac{\rho V D}{\mu} = \frac{(680)(1.06)(0.01)}{2.92\text{ E-4}} = 24{,}700 \qquad \frac{L}{D} = 500$$

The nine data points are plotted as the open circles in Fig. 5.10. The values of $L/D$ are listed for each point, and we see a significant length effect. In fact, if we connect the only two points which have the same $L/D$ ( = 200), we could see (and cross-plot to verify) that $\Delta p$ increases linearly with $L$, as stated in the last part of the problem. Since $L$ occurs only in $\Pi_3 = L/D$, the function $\Pi_1 = \text{fcn}(\Pi_2, \Pi_3)$ must reduce to $\Pi_1 = (L/D)\,\text{fcn}(\Pi_2)$, or simply a function involving only *two* parameters:

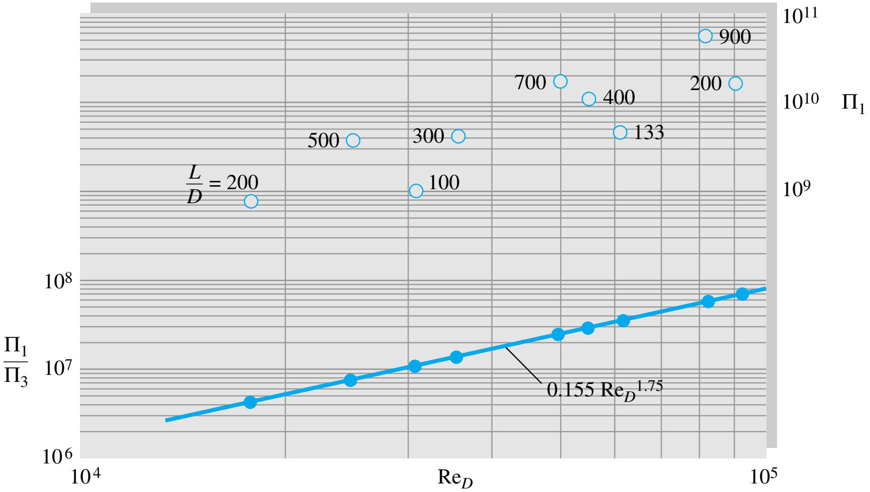



**Fig. 5.10** Two different correlations of the data in Example 5.7: Open circles when plotting $\rho D^2\,\Delta p/\mu^2$ versus $Re_D$, $L/D$ is a parameter; once it is known that $\Delta p$ is proportional to $L$, a replot (solid circles) of $\rho D^3\,\Delta p/(L\mu^2)$ versus $Re_D$ collapses into a single power-law curve.

$$\frac{\rho D^3\,\Delta p}{L\mu^2} = \text{fcn}\left(\frac{\rho VD}{\mu}\right) \qquad \text{flow in a long pipe} \qquad \textit{Ans. (c)}$$

We now modify each data point in Fig. 5.10 by dividing it by its $L/D$ value. For example, for the first row of data, $\rho D^3\,\Delta p/(L\mu^2) = (3.73\text{ E9})/500 = 7.46\text{ E6}$. We replot these new data points as solid circles in Fig. 5.10. They correlate almost perfectly into a straight-line power-law function:

$$\frac{\rho D^3\,\Delta p}{L\mu^2} \approx 0.155\left(\frac{\rho VD}{\mu}\right)^{1.75} \qquad \textit{Ans. (c)}$$

All newtonian smooth pipe flows should correlate in this manner. This example is a variation of the first completely successful dimensional analysis, pipe-flow friction, performed by Prandtl's student Paul Blasius, who published a related plot in 1911. For this range of (turbulent-flow) Reynolds numbers, the pressure drop increases approximately as $V^{1.75}$.

## EXAMPLE 5.8

The smooth-sphere data plotted in Fig. 5.3*a* represent dimensionless drag versus dimensionless *viscosity*, since ($\rho$, $V$, $d$) were selected as scaling or repeating variables. (*a*) Replot these data to display the effect of dimensionless *velocity* on the drag. (*b*) Use your new figure to predict the terminal (zero-acceleration) velocity of a 1-cm-diameter steel ball (SG = 7.86) falling through water at 20°C.

### Solution

To display the effect of velocity, we must not use $V$ as a repeating variable. Instead we choose ($\rho$, $\mu$, $d$) as our $j$ variables to nondimensionalize Eq. (5.1), $F = \text{fcn}(d, V, \rho, \mu)$. (See Example 5.2 for an alternate approach to this problem.) The pi groups form as follows:

$$\Pi_1 = \rho^a\mu^b d^c F = \frac{\rho F}{\mu^2} \qquad \Pi_2 = \rho^e\mu^f d^g V = \frac{\rho V d}{\mu} \qquad \textit{Ans. (a)}$$

That is, $a = 1$, $b = -2$, $c = 0$, $e = 1$, $f = -1$, and $g = 1$, by using our power-product techniques of Examples 5.2 to 5.6. Therefore a plot of $\rho F/\mu^2$ versus Re will display the direct effect of velocity on sphere drag. This replot is shown as Fig. 5.11. The drag increases rapidly with velocity up to transition, where there is a slight drop, after which it increases more quickly than ever. If the force is known, we may predict the velocity from the figure.

For water at 20°C, take $\rho = 998$ kg/m$^3$ and $\mu = 0.001$ kg/(m · s). For steel, $\rho_s = 7.86\rho_{water} \approx 7840$ kg/m$^3$. For terminal velocity, the drag equals the net weight of the sphere in water. Thus

$$F = W_{net} = (\rho_s - \rho_w)g\frac{\pi}{6}d^3 = (7840 - 998)(9.81)\left(\frac{\pi}{6}\right)(0.01)^3 = 0.0351 \text{ N}$$

Therefore the ordinate of Fig. 5.11 is known:

Falling steel sphere: $$\frac{\rho F}{\mu^2} = \frac{(998 \text{ kg/m}^3)(0.0351 \text{ N})}{[0.001 \text{ kg/(m}\cdot\text{s)}]^2} \approx 3.5\text{ E7}$$

From Fig. 5.11, at $\rho F/\mu^2 \approx 3.5$ E7, a magnifying glass reveals that $\text{Re}_d \approx 2$ E4. Then a crude estimate of the terminal fall velocity is

$$\frac{\rho V d}{\mu} \approx 20{,}000 \quad \text{or} \quad V \approx \frac{20{,}000[0.001 \text{ kg/(m}\cdot\text{s)}]}{(998 \text{ kg/m}^3)(0.01 \text{ m})} \approx 2.0\frac{\text{m}}{\text{s}}$$ *Ans. (b)*

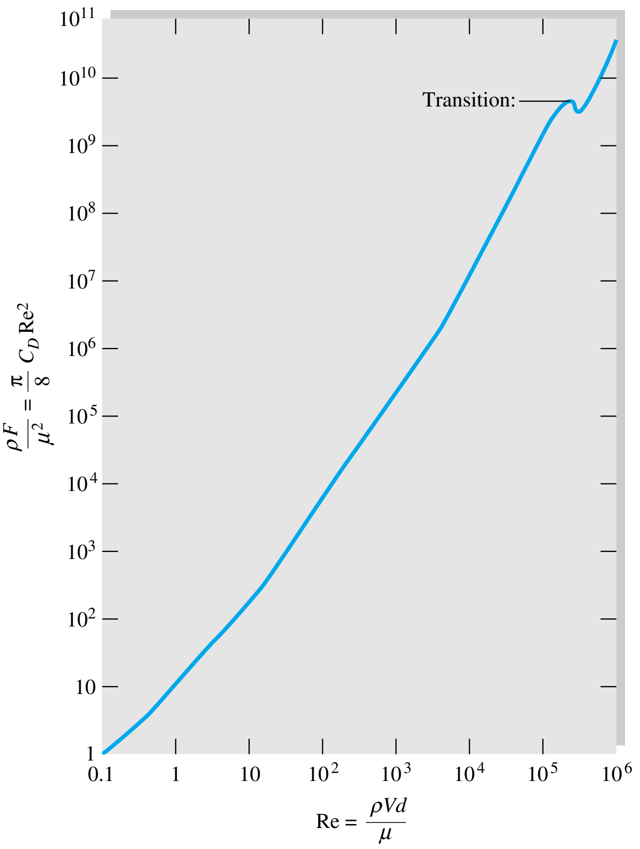



**Fig. 5.11** Cross-plot of sphere-drag data from Fig. 5.3*a* to isolate diameter and velocity.

Better accuracy could be obtained by expanding the scale of Fig. 5.11 in the region of the given force coefficient. However, there is considerable uncertainty in published drag data for spheres, so the predicted fall velocity is probably uncertain by at least ±5 percent.

Note that we found the answer directly from Fig. 5.11. We could use Fig. 5.3*a* also but would have to iterate between the ordinate and abscissa to obtain the final result, since *V* is contained in both plotted variables.

## Summary

Chapters 3 and 4 presented integral and differential methods of mathematical analysis of fluid flow. This chapter introduces the third and final method: experimentation, as supplemented by the technique of dimensional analysis. Tests and experiments are used both to strengthen existing theories and to provide useful engineering results when theory is inadequate.

The chapter begins with a discussion of some familiar physical relations and how they can be recast in dimensionless form because they satisfy the principle of dimensional homogeneity. A general technique, the pi theorem, is then presented for systematically finding a set of dimensionless parameters by grouping a list of variables which govern any particular physical process. Alternately, direct application of dimensional analysis to the basic equations of fluid mechanics yields the fundamental parameters governing flow patterns: Reynolds number, Froude number, Prandtl number, Mach number, and others.

It is shown that model testing in air and water often leads to scaling difficulties for which compromises must be made. Many model tests do not achieve true dynamic similarity. The chapter ends by pointing out that classic dimensionless charts and data can be manipulated and recast to provide direct solutions to problems that would otherwise be quite cumbersome and laboriously iterative.

## Problems

Most of the problems herein are fairly straightforward. More difficult or open-ended assignments are labeled with an asterisk. Problems labeled with an EES icon, for example, Prob. 5.61, will benefit from the use of the Engineering Equation Solver (EES), while problems labeled with a computer icon may require the use of a computer. The standard end-of-chapter problems 5.1 to 5.91 (categorized in the problem list below) are followed by word problems W5.1 to W5.10, fundamentals of engineering exam problems FE5.1 to FE5.10, comprehensive applied problems C5.1 to C5.4, and design projects D5.1 and D5.2.

**Problem distribution**

| Section | Topic | Problems |
|---|---|---|
| 5.1 | Introduction | 5.1–5.6 |
| 5.2 | Choosing proper scaling parameters | 5.7–5.9 |
| 5.2 | The principle of dimensional homogeneity | 5.10–5.17 |
| 5.3 | The pi theorem | 5.18–5.41 |
| 5.4 | Nondimensionalizing the basic equations | 5.42–5.47 |
| 5.4 | Data for spheres and cylinders | 5.48–5.57 |
| 5.5 | Scaling of model data | 5.58–5.74 |
| 5.5 | Froude- and Mach-number scaling | 5.75–5.84 |
| 5.5 | Inventive rescaling of the data | 5.85–5.91 |

**P5.1** For axial flow through a circular tube, the Reynolds number for transition to turbulence is approximately 2300 [see Eq. (6.2)], based upon the diameter and average velocity. If $d = 5$ cm and the fluid is kerosine at 20°C, find the volume flow rate in $m^3$/h which causes transition.

**P5.2** In flow past a thin flat body such as an airfoil, transition to turbulence occurs at about Re = 1 E6, based on the distance *x* from the leading edge of the wing. If an airplane flies at 450 mi/h at 8-km standard altitude and undergoes transition at the 12 percent chord position, how long is its chord (wing length from leading to trailing edge)?