CONDUCTION HEAT TRANSFER

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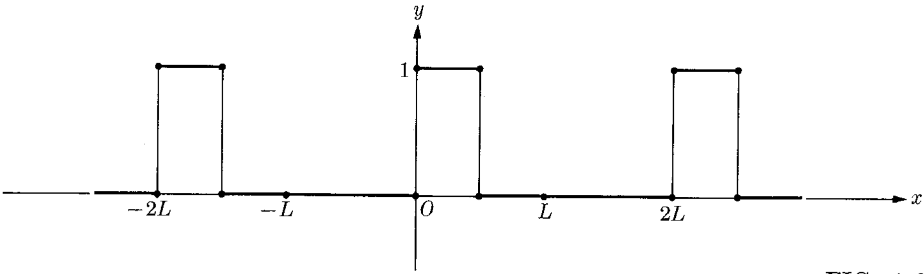

FIG. 4–9

Having gained the foregoing mathematical background we now proceed to the solution of problems by the method of separation of variables.

4–5. Separation of Variables. Steady Two-Dimensional Cartesian Geometry

When the boundary conditions of a problem are in terms of specified T, $\partial T/\partial n$, or $\partial T/\partial n + BT$, where n is the normal to the boundary and B a constant, the solution may be expressed as a product of functions of each coordinate separately. This allows the boundary conditions to be expressed in terms of a single variable, and reduces the partial differential equation to a set of ordinary differential equations.

The essential features of the method will now be illustrated by means of a steady two-dimensional example. Consider the second-order partial differential equation

$$a_1(x)\,\frac{\partial^2 T}{\partial x^2} + a_2(x)\,\frac{\partial T}{\partial x} + a_3(x)T + b_1(y)\,\frac{\partial^2 T}{\partial y^2} + b_2(y)\,\frac{\partial T}{\partial y} + b_3(y)T = 0. \tag{4–41}$$

A more generalized form of this equation which involves coefficients as functions of both independent variables is not suitable for the separation of variables.

Assume the existence of a product solution

$$T(x, y) = X(x)Y(y), \tag{4–42}$$

where X is a function of x alone and Y is a function of y. This assumption becomes meaningful when the two functions X and Y actually satisfy separate differential equations.

Introducing Eq. (4–42) into Eq. (4–41) and dividing the result by XY yields

$$\left[a_1(x)\,\frac{d^2X}{dx^2} + a_2(x)\,\frac{dX}{dx} + a_3X\right]\frac{1}{X} = -\left[b_1(y)\,\frac{d^2Y}{dy^2} + b_2(y)\,\frac{dY}{dy} + b_3(y)\,Y\right]\frac{1}{Y}\cdot \tag{4–43}$$

The left-hand side of this equation is independent of y and the right-hand side is independent of x. Since x and y can vary independently, both sides of Eq. (4-43) must be independent of either variable; that is, they must be equal to a constant, say $+\lambda^2$ or $-\lambda^2$. This constant is called the *separation parameter.* Hence the partial differential equation of Eq. (4-41) is reduced to the following two ordinary differential equations:

$$
\begin{aligned}
a_1(x)\frac{d^2X}{dx^2} + a_2(x)\frac{dX}{dx} + [a_3(x) \pm \lambda^2]X = 0,\\
b_1(y)\frac{d^2Y}{dy^2} + b_2(y)\frac{dY}{dy} + [b_3(y) \mp \lambda^2]Y = 0.
\end{aligned}
\tag{4-44}
$$

The method of separation of variables is applicable to steady two-dimensional problems if and when (i) *one of the directions of the problem is expressed by a homogeneous differential equation subject to homogeneous boundary conditions (the homogeneous direction), while the other direction is expressed by a homogeneous differential equation subject to one homogeneous and one nonhomogeneous boundary condition (the nonhomogeneous direction),* and (ii) *the sign of* λ^2 *is chosen such that the boundary-value problem of the homogeneous direction leads to a characteristic-value problem.*

The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively. This chapter and the next one are devoted to series solutions which are applicable to problems homogeneous in finite directions. Problems requiring homogeneity in infinite domains are suitable to integral solutions. These, being easier to solve by the Laplace transforms, are delayed to Chapter 7 (see Section 7-5).

The result of the present section may readily be extended to steady three-dimensional problems and to unsteady problems (see Section 4-11 and Chapter 5, respectively). We shall now illustrate the method of separation of variables by a number of examples.

Example 4-1. Consider an infinitely long two-dimensional fin of thickness l (Fig. 4-10). The base temperature of the fin is $F(y)$, the ambient temperature T_∞. The heat transfer coefficient is large. We wish to find the steady temperature of the fin.

The differential formulation of the problem, according to the selected reference frame, is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

$$T(0, y) = F(y), \qquad T(\infty, y) = T_\infty,$$

$$T(x, 0) = T_\infty, \qquad T(x, l) = T_\infty.$$

We now seek a solution by the method of separation of variables, which requires that the differential equation and three of the boundary conditions be homogeneous. Although the problem expressed in T does not satisfy these conditions, the simple transformation

$$\theta = T - T_\infty$$

reduces three of the nonhomogeneous boundary conditions to homogeneous conditions without affecting the homogeneity of the differential equation. Thus the formulation of the problem in θ becomes

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \qquad (4\text{–}45)$$

$$\theta(0, y) = F(y) - T_\infty = f(y), \qquad (4\text{–}46)$$

$$\theta(\infty, y) = 0, \qquad (4\text{–}47)$$

$$\theta(x, 0) = 0, \qquad (4\text{–}48)$$

$$\theta(x, l) = 0. \qquad (4\text{–}49)$$

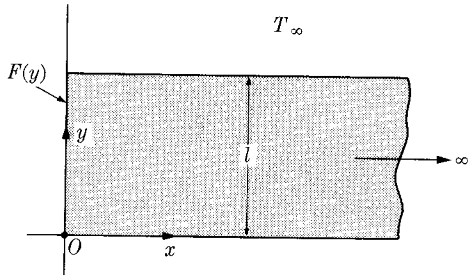

FIG. 4–10

Assuming the existence of a product solution of the form

$$\theta(x, y) = X(x)Y(y), \qquad (4\text{–}50)$$

then introducing Eq. (4–50) into Eq. (4–45) and dividing each term by XY, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \pm\lambda^2. \qquad (4\text{–}51)$$

Here the sign of λ^2 must be chosen such that the homogeneous y-direction results in a characteristic-value problem. The selection of $-\lambda^2$ yields particular solutions in y expressible by hyperbolic functions which, as indicated in Section 4–1, cannot be made orthogonal; hence $+\lambda^2$ is suitable to our problem.

Further use of Eq. (4–50) reduces the two-dimensional homogeneous boundary conditions of the problem to one-dimensional conditions. This may readily be illustrated using one of these conditions, say Eq. (4–47). Thus, since $Y(y)$ is arbitrary, $\theta(\infty, y) = X(\infty)Y(y) = 0$ implies $X(\infty) = 0$.

Finally, we have the problem separately expressed in the x- and y-directions as follows:

$$\frac{d^2Y}{dy^2} + \lambda^2 Y = 0; \qquad Y(0) = 0, \quad Y(l) = 0, \qquad (4\text{–}52)$$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0; \qquad X(\infty) = 0. \qquad (4\text{–}53)$$