2.3
The Heat Diffusion Equation

A major objective in a conduction analysis is to determine the *temperature field* in a medium resulting from conditions imposed on its boundaries. That is, we wish to know the *temperature distribution*, which represents how temperature varies with position in the medium. Once this distribution is known, the conduction heat flux at any point in the medium or on its surface may be computed from Fourier's law. Other important quantities of interest may also be determined. For a solid, knowledge of the temperature distribution could be used to ascertain structural integrity through determination of thermal stresses, expansions, and deflections. The temperature distribution could also be used to optimize the thickness of an insulating material or to determine the compatibility of special coatings or adhesives used with the material.

We now consider the manner in which the temperature distribution can be determined. The approach follows the methodology described in Section 1.3.3 of applying the energy conservation requirement. In this case, we define a *differential control volume*, identify the relevant energy transfer processes, and introduce the appropriate rate equations. The result is a differential equation whose solution, for prescribed boundary conditions, provides the temperature distribution in the medium.

Consider a homogeneous medium within which there is no bulk motion (advection) and the temperature distribution $T(x, y, z)$ is expressed in Cartesian coordinates. Following the methodology of applying conservation of energy (Section 1.3.3), we first define an infinitesimally small (differential) control volume, $dx \cdot dy \cdot dz$, as shown in Figure 2.11. Choosing to formulate the first law at an instant of time, the second step is to consider the energy processes that are relevant to this control volume. If there are temperature gradients, conduction heat transfer will occur across

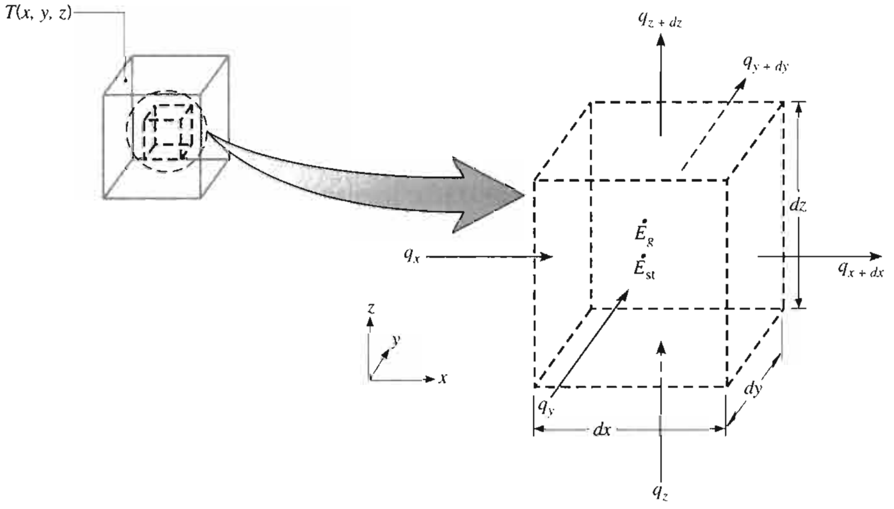

FIGURE 2.11 Differential control volume, $dx\, dy\, dz$, for conduction analysis in Cartesian coordinates.

each of the control surfaces. The conduction heat rates perpendicular to each of the control surfaces at the x, y, and z coordinate locations are indicated by the terms q_x, q_y, and q_z, respectively. The conduction heat rates at the opposite surfaces can then be expressed as a Taylor series expansion where, neglecting higher order terms,

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx \tag{2.11a}$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy \tag{2.11b}$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz \tag{2.11c}$$

In words, Equation 2.11a simply states that the x component of the heat transfer rate at $x + dx$ is equal to the value of this component at x plus the amount by which it changes with respect to x times dx.

Within the medium there may also be an *energy source* term associated with the rate of thermal energy generation. This term is represented as

$$\dot{E}_g = \dot{q}\, dx\, dy\, dz \tag{2.12}$$

where $\dot{q}$ is the rate at which energy is generated per unit volume of the medium (W/m^3). In addition, there may occur changes in the amount of the internal thermal energy stored by the material in the control volume. If the material is not experiencing a change in phase, latent energy effects are not pertinent, and the *energy storage* term may be expressed as

$$\dot{E}_{\text{st}} = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz \tag{2.13}$$

where $\rho c_p\, \partial T/\partial t$ is the time rate of change of the sensible (thermal) energy of the medium per unit volume.

Once again it is important to note that the terms $\dot{E}_g$ and $\dot{E}_{\text{st}}$ represent different physical processes. The energy generation term $\dot{E}_g$ is a manifestation of some energy conversion process involving thermal energy on one hand and some other form of energy, such as chemical, electrical, or nuclear, on the other. The term is positive (a *source*) if thermal energy is being generated in the material at the expense of some other energy form; it is negative (a *sink*) if thermal energy is being consumed. In contrast, the energy storage term $\dot{E}_{\text{st}}$ refers to the rate of change of thermal energy stored by the matter.

The last step in the methodology outlined in Section 1.3.3 is to express conservation of energy using the foregoing rate equations. On a *rate* basis, the general form of the conservation of energy requirement is

$$\dot{E}_{\text{in}} + \dot{E}_g - \dot{E}_{\text{out}} = \dot{E}_{\text{st}} \tag{1.11c}$$

Hence, recognizing that the conduction rates constitute the energy inflow, $\dot{E}_{\text{in}}$, and outflow, $\dot{E}_{\text{out}}$, and substituting Equations 2.12 and 2.13, we obtain

$$q_x + q_y + q_z + \dot{q}\, dx\, dy\, dz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz \tag{2.14}$$

Substituting from Equations 2.11, it follows that

$$-\frac{\partial q_x}{\partial x}dx - \frac{\partial q_y}{\partial y}dy - \frac{\partial q_z}{\partial z}dz + \dot{q}\,dx\,dy\,dz = \rho c_p \frac{\partial T}{\partial t}dx\,dy\,dz \tag{2.15}$$

The conduction heat rates may be evaluated from Fourier's law,

$$q_x = -k\,dy\,dz\frac{\partial T}{\partial x} \tag{2.16a}$$

$$q_y = -k\,dx\,dz\frac{\partial T}{\partial y} \tag{2.16b}$$

$$q_z = -k\,dx\,dy\frac{\partial T}{\partial z} \tag{2.16c}$$

where each heat flux component of Equation 2.6 has been multiplied by the appropriate control surface (differential) area to obtain the heat transfer rate. Substituting Equations 2.16 into Equation 2.15 and dividing out the dimensions of the control volume (*dx dy dz*), we obtain

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t} \tag{2.17}$$

Equation 2.17 is the general form, in Cartesian coordinates, of the *heat diffusion equation.* This equation, often referred to as the *heat equation*, provides the basic tool for heat conduction analysis. From its solution, we can obtain the temperature distribution $T(x, y, z)$ as a function of time. The apparent complexity of this expression should not obscure the fact that it describes an important physical condition, that is, conservation of energy. You should have a clear understanding of the physical significance of each term appearing in the equation. For example, the term $\partial(k\partial T/\partial x)/\partial x$ is related to the *net* conduction heat flux *into* the control volume for the x-coordinate direction. That is, multiplying by dx,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx = q''_x - q''_{x+dx} \tag{2.18}$$

with similar expressions applying for the fluxes in the y and z directions. In words, the heat equation, Equation 2.17, therefore states that *at any point in the medium the net rate of energy transfer by conduction into a unit volume plus the volumetric rate of thermal energy generation must equal the rate of change of thermal energy stored within the volume.*

It is often possible to work with simplified versions of Equation 2.17. For example, if the thermal conductivity is constant, the heat equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.19}$$

where $\alpha = k/\rho c_p$ is the *thermal diffusivity.* Additional simplifications of the general form of the heat equation are often possible. For example, under *steady-state* y dz (2.15)

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conditions, there can be no change in the amount of energy storage; hence Equation 2.17 reduces to

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0 \tag{2.20}$$

Moreover, if the heat transfer is *one-dimensional* (e.g., in the x direction) and there is *no energy generation*, Equation 2.20 reduces to

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \tag{2.21}$$

The important implication of this result is that *under steady-state, one-dimensional conditions with no energy generation*, the heat flux is a constant in the direction of transfer ($dq''_x/dx = 0$).

The heat equation may also be expressed in cylindrical and spherical coordinates. The differential control volumes for these two coordinate systems are shown in Figures 2.12 and 2.13.

Cylindrical Coordinates When the del operator ∇ of Equation 2.3 is expressed in cylindrical coordinates, the general form of the heat flux vector, and hence of Fourier's law, is

$$\boldsymbol{q}''=-k\nabla T=-k\left(\boldsymbol{i}\frac{\partial T}{\partial r}+\boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \phi}+\boldsymbol{k}\frac{\partial T}{\partial z}\right) \tag{2.22}$$

where

$$q''_r=-k\frac{\partial T}{\partial r}\qquad q''_\phi=-\frac{k}{r}\frac{\partial T}{\partial \phi}\qquad q''_z=-k\frac{\partial T}{\partial z} \tag{2.23}$$

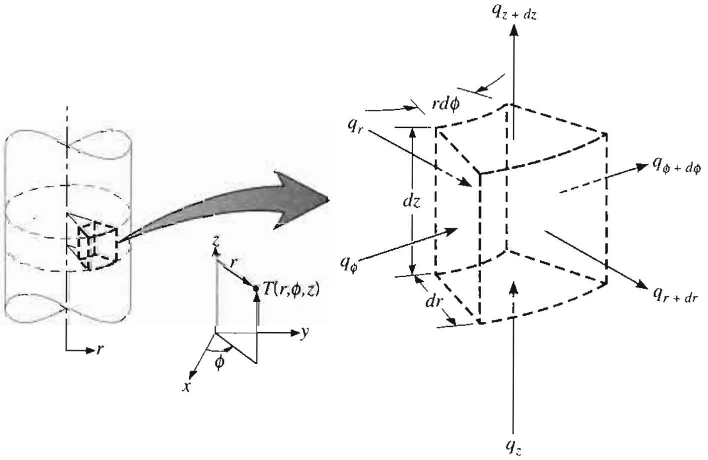

FIGURE 2.12 Differential control volume, $dr\cdot r\,d\phi\cdot dz$, for conduction analysis in cylindrical coordinates (r, ϕ, z).

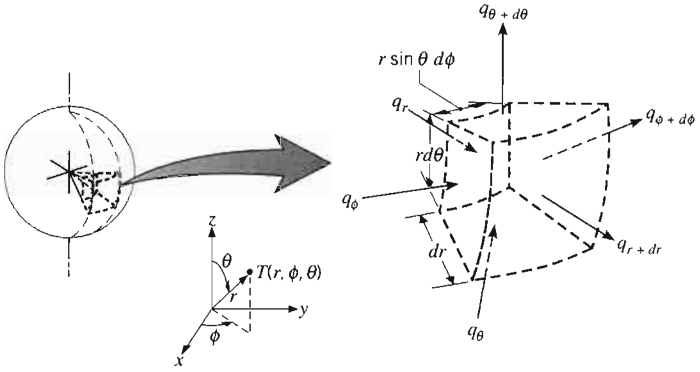

FIGURE 2.13 Differential control volume, $dr \cdot r\sin\theta\,d\phi \cdot r\,d\theta$, for conduction analysis in spherical coordinates (r, ϕ, θ).

are heat flux components in the radial, circumferential, and axial directions, respectively. Applying an energy balance to the differential control volume of Figure 2.12, the following general form of the heat equation is obtained:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t} \tag{2.24}$$

Spherical Coordinates In spherical coordinates the general form of the heat flux vector and Fourier's law is

$$\mathbf{q}'' = -k\nabla T = -k\left(\mathbf{i}\frac{\partial T}{\partial r} + \mathbf{j}\frac{1}{r}\frac{\partial T}{\partial \theta} + \mathbf{k}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\right) \tag{2.25}$$

where

$$q''_r = -k\frac{\partial T}{\partial r} \qquad q''_\theta = -\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad q''_\phi = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi} \tag{2.26}$$

are heat flux components in the radial, polar, and azimuthal directions, respectively. Applying an energy balance to the differential control volume of Figure 2.13, the following general form of the heat equation is obtained:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t} \tag{2.27}$$

Since it is important that you be able to apply conservation principles to differential control volumes, you should attempt to derive Equation 2.24 or 2.27 (see Problems 2.35 and 2.36). Note that the temperature gradient in Fourier's law must have units of K/m. Hence, when evaluating the gradient for an angular coordinate, it must be expressed in terms of the differential change in arc *length*. For example, the heat flux component in the circumferential direction of a cylindrical coordinate system is $q''_\phi = -(k/r)(\partial T/\partial \phi)$, and *not* $q''_\phi = -k(\partial T/\partial \phi)$.

EXAMPLE 2.2

The temperature distribution across a wall 1 m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in degrees Celsius and x is in meters, while $a = 900°\text{C}$, $b = -300°\text{C/m}$, and $c = -50°\text{C/m}^2$. A uniform heat generation, $\dot{q} = 1000\ \text{W/m}^3$, is present in the wall of area $10\ \text{m}^2$ having the properties $\rho = 1600\ \text{kg/m}^3$, $k = 40\ \text{W/m}\cdot\text{K}$, and $c_p = 4\ \text{kJ/kg}\cdot\text{K}$.

1. Determine the rate of heat transfer entering the wall ($x = 0$) and leaving the wall ($x = 1$ m).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at $x = 0$, 0.25, and 0.5 m.

SOLUTION

Known: Temperature distribution $T(x)$ at an instant of time t in a one-dimensional wall with uniform heat generation.

Find:

1. Heat rates entering, q_{in} ($x = 0$), and leaving, q_{out} ($x = 1$ m), the wall.
2. Rate of change of energy storage in the wall, $\dot{E}_{\text{st}}$.
3. Time rate of temperature change at $x = 0$, 0.25, and 0.5 m.

Schematic: